Rev. Téc. Ing. Univ. Zulia. Vol. 33, N° 1, 87 - 93, 2010

A summation formula concerning the Mellin Transform*

M.L. Glasser[1,2], Nikos Bagis[3], Grazyna Majchrowska[3]

[1]Department of Physics, Clarkson University, Potsdam. New York, USA. laryg@clarkson.edu
[2]Departamento de Física Teórica, Atómica y Óptica, Universidad de Valladolid, 47071. Spain.
[3]Department of Informatics, Aristotele University of Thessaloniki 54124, Greece.

Abstract

We derive a new summation formula involving Mellin transform. We apply this formula to sum various series related to elliptic integrals and theta functions.

Key words: Mellin Transform, summation formulas, Jacobian functions.

Una fórmula de adición usando la Transformada de Mellin

Resumen

En este trabajo una nueva fórmula de adición es derivada usando la transformada de Mellin. Esta fórmula es aplicada para sumar varias series relacionadas con las integrales elípticas y las funciones theta.

Palabra clave: Transformada de Mellin, fórmulas de adición, función jacobiana.

1. Introduction

One of the fundamental results of Fourier analysis is the Poisson Summation Formula, which can be written [1]

$$a \sum_{k=-\infty}^{\infty} f(ak) = \sum_{k=-\infty}^{\infty} \hat{f}(bk) \qquad (1)$$

where $a > 0$, $ab = 2\pi$ and

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(t) e^{-itx} dt \qquad (2)$$

is the Fourier Transform of f. Equation (1) is valid under relatively weak conditions. For example $f(x) = O\left[(1+|x|^c)^{-1}\right]$, for $x \in \mathbf{R}$ and for some $c > 0$. It is also known that if

$$\hat{f}_c(y) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \cos(ty) dt \qquad (3)$$

is the Cosine Transform of a function f then we have the Poisson Summation Cosine Formula [2]

$$\sqrt{a}\left(\frac{f(0)}{2} + \sum_{k=1}^{\infty} f(ka)\right) = \sqrt{b}\left(\frac{\hat{f}_c(0)}{2} + \sum_{k=1}^{\infty} \hat{f}_c(kb)\right) \qquad (4)$$

where $a > 0$, $ab = 2\pi$.

In this work using the Poisson Summation Formula we generalize an exponential formula of Ramanujan and arrive at a new Mellin Summation Formula. This formula is like (4) but now the part of Fourier Cosine Transform is replaced by the Mellin Transform. We also give several applications.

Definition [2]

The Mellin Transform of a function Ψ is defined to be

$$(M\Psi)(z) := \int_0^{\infty} \Psi(t) t^{z-1} dt \qquad (5)$$

* Mathematics Subject Classification: 33E05, 44A15, 44A20.

For the applications we will need the following

2. The Theorems

Theorem 1 [3]

Let $a > 0$ and the function f be odd and analytic in the uper half plane $\operatorname{Im}(z) > 0$ and continuous in $\operatorname{Im}(z) \geq 0$ and let there exist $C, N > 0$ and $0 \leq b < \pi$ such that

$$|f(z)| \leq C(1+|z|^N)e^{b|Re(z)|} \quad (6)$$

for every z in $\operatorname{Im}(z) \geq 0$, then

$$\sqrt{a}\left(\frac{f'(0)}{2\pi} + \sum_{k=1}^{\infty}\frac{f(ka)}{\sinh(\pi ka)}\right) =$$

$$\sqrt{\frac{2b}{\pi}}\left(\frac{c_o}{2} + i\sum_{k=1}^{\infty}\frac{(-1)^k f(ki)}{e^{bk}-1}\right) \quad (7)$$

where $ab = 2\pi$ and $c_o = \lim_{x\to 0^+}\sum_{k=1}^{\infty}(-1)^k if(ik)e^{-kx}$.

The Main Theorem.
(The Mellin Summation Formula-MSF)

Let $a, b > 0$ and $ab = 2\pi$, then

$$a\lim_{r\to 1^-}\left(\sum_{k=1}^{\infty} y(k)f(k)r^k\right) + a\sum_{k=1}^{\infty}\frac{y(k)f(k)}{e^{ka}-1} +$$

$$a\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty} y(k)f(k)e^{kna}\right) = c + 2\sum_{k=1}^{\infty}\phi(kb) \quad (8)$$

where $\phi(x) = Re[(M\Psi)(ix)f(-ix)]$, $c = \lim_{h\to 0} Re[(M\Psi)(ih)f(-ih)]$ with $\Psi(x) = \sum_{k=0}^{\infty} y(k)x^k$ and f, ψ, x real, $f(0) = 0$, provided that all sums converge.

To prove the MSF we use a Theorem which appeared first in [4]. Here we give a complete proof and the conditions under which this Theorem holds.

Theorem 2. [4]

Let $\Psi(x)$ be analytic around 0. Also let f be analytic function in $\mathbf{C}$ satisfying

$$|f(z)(M\Psi)(x+iz)| \leq C(1+|z|)^{\lambda}e^{-\delta|Re(z)|} \quad (9)$$

for every z with $\operatorname{Im}(z) \geq 0$, $C, \lambda, \delta > 0$ constantst, with the condition $|z| = x + N + 1/2$, N sufficiently large natural number. Then for $x > 0$ the integral

$$\int_{-\infty}^{\infty} f(t)(M\Psi)(x+it)dt$$

converges absolutely, the series

$$\sum_{m=0}^{\infty}\frac{\Psi^{(m)}(0)}{m!}f(i(x+m))$$

converges in the Abel sense and

$$\int_{-\infty}^{\infty} f(t)(M\Psi)(x+it)dt =$$

$$2\pi\lim_{r\to 1^-}\sum_{m=0}^{\infty}\frac{\Psi^{(m)}(0)}{m!}f(i(x+m))r^m \quad (10)$$

If also

$$\left|\frac{\Psi^{(m)}(0)}{m!}f(i(x+m))\right| \leq \frac{C'}{m+1}$$

then the series

$$\sum_{m=0}^{\infty}\frac{\Psi^{(m)}(0)}{m!}f(i(x+m))$$

converges and we can drop the limit in (10).

3. The Proofs of Theorems

The proof of Theorem 1 is in [3]. For the proof of Theorem 2 we need a Lemma.

Lemma

Let Ψ have a Taylor Series around 0, with radius of convergence $r > 0$. Let also $x \in \mathbf{R}$ such that

$$\int_0^{\infty}|\Psi(u)|u^{x-1}du < +\infty \quad (11)$$

Then the Mellin Transform of Ψ can be extended analytically into a meromorphic function in the half plane $Re(z) < x$ with simple poles at the points $z = -m$ for $m \in \mathbf{Z}$ with $m > -x$, $m \geq 0$.

Proof

Let $0 < a < r$. Then if z is not an integer

$$\int_0^a \Psi(u)u^{z-1}du = \int_0^a \sum_{m=0}^{\infty} \frac{\Psi^{(m)}(0)}{m!} u^{z+m-1}du =$$
$$\sum_{m=0}^{\infty} \int_0^a u^{z+m-1}du \frac{\Psi^{(m)}(0)}{m!} = \sum_{m=0}^{\infty} \frac{\Psi^{(m)}(0)}{m!} \frac{a^{z+m}}{z+m}$$

Thus the function $h_1(z) = \int_0^a \Psi(u)u^{z-1}du$ is meromorphic in $\mathbf{C}$ whose only poles are $z = -m$ with residue $\frac{\Psi^{(m)}(0)}{m!}$.

We also define the function $h_2(z) = \int_a^{\infty} \Psi(u)u^{z-1}du$ which converges absolutely to ananalytic function when $Re(z) < x$.

$$\int_a^{\infty} \left|\Psi(u)u^{z-1}\right| du = \int_a^{\infty} |\Psi(u)| u^{Re(z)-1}du \leq$$
$$a^{Re(a)-x} \int_a^{\infty} |\Psi(u)| u^{x-1}du < +\infty$$

And for the derivative we have

$$\int_a^{\infty} \left|\Psi(u)u^{z-1} \log(u)\right| \leq C_z \int_a^{\infty} |\Psi(u)| u^{x-1}du < +\infty,$$

$(Re(z) < x)$.

This completes the proof of the Lemma.

Now let the function Ψ be as in Lemma and let $x \in \mathbf{R}$ be such that (11) holds. We define the function

$$g(z) = f(z)(M\Psi)(x + iz) \tag{12}$$

Then from the Lemma g is meromorphic in $\text{Im}(z) > 0$ and continuous in $\text{Im}(z) \geq 0$ with simple poles at $z = i(m + x)$, where $m \in Z$, $m \geq 0$, $m + x > 0$ and

$$Res\big(g, i(x + m)\big) = \frac{\Psi^{(m)}(0)}{im!} f\big(i(x + m)\big) \tag{13}$$

Let γ_R be the upper half circle with diameter $[-R, R]$ where $R = R_N = x + N + 1/2$, N a natural number. Then

$$\frac{1}{2\pi i} \int_{\gamma_R} g(z)dz = \sum_{i(x+m)-inside\gamma_R} \frac{\Psi^{(m)}(0)}{im!} f\big(i(x + m)\big) \tag{14}$$

and thus

$$\int_{-R_N}^{R_N} f(t)(M\Psi)(x + it)dt +$$
$$\int_0^{\pi} f(R_N e^{i\theta})(M\Psi)(x + iR_N e^{i\theta})iR_N e^{i\theta}d\theta =$$
$$2\pi \sum_{0 \leq m \leq N, m > -x} \frac{\Psi^{(m)}(0)}{m!} f\big(i(x + m)\big)$$

So, if the integral $\int_{-\infty}^{\infty} f(t)(M\Psi)(x + it)dt$ exists and also

$$\lim_{N \to \infty} R_N \left| \int_0^{\pi} f(R_N e^{i\theta})(M\Psi)(x + iR_N e^{i\theta})d\theta \right| = 0 \tag{15}$$

the series converges and we have

$$\int_{-\infty}^{\infty} f(t)(M\Psi)(x + it)dt =$$
$$2\pi \sum_{m=0, m > -x}^{\infty} \frac{\Psi^{(m)}(0)}{m!} f\big(i(x + m)\big)$$

The condition $m > -x$ not needed if $x > 0$.

Having in mind the above and the Lemma we can proceed to the following

Proof of Theorem 2

From (9) the integral

$$\int_{-\infty}^{\infty} f(t)(M\Psi)(x + it)e^{ita}dt$$

is absolutely convergent for $a > 0$. Also if $0 < a \leq \delta$ we will have

$$R_N \left| f(R_N e^{i\theta})(M\Psi)(x + iR_N e^{i\theta}) \right| \left| e^{iaR_N e^{i\theta}} \right|$$
$$\leq C(1 + R_N)^{\lambda+1} e^{-\delta R_N |\cos(\theta)|} e^{-aR_N \sin(\theta)}$$
$$\leq C(1 + R_N)^{\lambda+1} e^{-aR_N} \to 0$$

for $R_N \to \infty$ when $0 \leq \theta \leq \pi$.

Hence for every $a > 0$ one as

$$\int_{-\infty}^{\infty} f(t)(M\Psi)(x+it)e^{ita}dt =$$

$$2\pi \sum_{m=0}^{\infty} \frac{\Psi^{(m)}(0)}{m!} f(i(x+m))e^{-a(x+m)} \quad (16)$$

from which (10) follows.

Proof of the Main Theorem

If $\phi(x) = Re\left[(M\Psi)(ix)f(-ix)\right]$, with f, y, x real, then $\phi(x)$ is an even function. The reason is that we can write according Theorem 2

$$\int_{-\infty}^{\infty} \phi(x)e^{ixa}dx = \int_{-\infty}^{\infty} Re\left[(M\Psi)(ix)f(-ix)\right]e^{ixa}dx$$

$$= \int_{-\infty}^{\infty} \frac{(M\Psi)(ix)f(-ix) + (M\Psi)(-ix)f(ix)}{2} e^{ixa}dx$$

$$= \int_{-\infty}^{\infty} \frac{M\Psi(ix)f(-ix)}{2} e^{ixa}dx + \int_{-\infty}^{\infty} \frac{M\Psi(-ix)f(ix)}{2} e^{ixa}dx$$

$$= \pi \sum_{k=0}^{\infty} \frac{\Psi^{(k)}(0)}{k!} f(k)e^{-ka} + \pi \sum_{k=0}^{\infty} \frac{\Psi^{(k)}(0)}{k!} f(k)e^{ka}$$

$$= 2\int_0^{\infty} \phi(x)\cos(xa)dx$$

From the above we have

$$\hat{\phi}_c(a) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \phi(x)\cos(xa)dx$$

$$= \sqrt{\frac{\pi}{2}} \sum_{k=0}^{\infty} \frac{\Psi^{(k)}(0)}{k!} f(k)e^{-ka} + \sqrt{\frac{\pi}{2}} \sum_{k=0}^{\infty} \frac{\Psi^{(k)}(0)}{k!} f(k)e^{ka}$$

also

$$\sqrt{b}\left(\frac{c}{2} + \sum_{k=0}^{\infty} \phi(kb)\right) =$$

$$\frac{\sqrt{a}}{2}\left(\sqrt{\frac{\pi}{2}} \sum_{k=1}^{\infty} \frac{\Psi^{(k)}(0)}{k!} f(k) + \sqrt{\frac{\pi}{2}} \sum_{k=1}^{\infty} \frac{\Psi^{(k)}(0)}{k!} f(k)\right)$$

$$+\sqrt{a}\sqrt{\frac{\pi}{2}} \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty} \frac{\Psi^{(k)}(0)}{k!} f(k)e^{-kna} + \sum_{k=1}^{\infty} \frac{\Psi^{(k)}(0)}{k!} f(k)e^{kna}\right)$$

$$= \sqrt{\frac{a\pi}{2}}\left(\sum_{k=1}^{\infty} \frac{\Psi^{(k)}(0)}{k!} f(k) + \sum_{k=1}^{\infty} \frac{\Psi^{(k)}(0)}{k!} \frac{f(k)}{e^{ka}-1} + \right.$$

$$\left.\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty} \frac{\Psi^{(k)}(0)}{k!} f(k)e^{kna}\right)\right)$$

where $ab = 2\pi$, $y(k) = \frac{\Psi^{(k)}(0)}{k!}$ and $c = \lim_{h\to 0} Re\left[(M\Psi)(ih)f(-ih)\right]$. This completes the proof.

Notes

1. In the same way one can prove a formula of Ramajunan [5] (which we give here a more general form)

If $ab = 2\pi$, then

$$a \sum_{k=0}^{\infty} \sum_{r=1}^{\infty} X(k)\exp(-re^{ak}) =$$

$$= a\left(\frac{L(0)}{2} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}L(-k)}{k!(e^{ka}-1)}\right) -$$

$$\gamma L(0) + L'(0) + 2\sum_{k=1}^{\infty} \phi(bk) \quad (17)$$

where $L(z) = \sum_{k=1}^{\infty} \frac{X(k)}{k^z}$ and $\phi(x) = \mathrm{Im}\left[\frac{\Gamma(ix+1)}{x} L(ix)\right]$.

2. The use of Theorem 2 for finding Self Reciprocal functions.

From [2] we have that whenever $\hat{f}_c(x) = f(x)$ then

$$(Mf)(s) = \sqrt{\frac{2}{\pi}} \Gamma(s)\cos\frac{\pi s}{2}(Mf)(1-s)$$

and

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^{s/2}\Gamma\left(\frac{s}{2}\right)\psi(s)x^{-s}ds$$

where $\psi(s) = f_e\left(s-\frac{1}{2}\right)$, $f_e(s)$ being an even function. Using Theorem 2 with $\Psi(x) = e^{-x}$ we have

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^{s/2}\Gamma\left(\frac{s}{2}\right)\psi(s)x^{-s}ds =$$

$$2\sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!} \psi(-2k)x^{2k}$$

which is more convenient to calculate.

Similar expansions hold for the Fourier sine transform and Hankel transforms.

4. Applications

1. Let $y(k) = \frac{(-1)^k}{k!}$ and $f(k) = kn^k$, then $\Psi(x) = e^{-x}$ and $(M\Psi)(s) = \Gamma(s)$. Hence if $ab = 2\pi$

$$1 - an\sum_{k=0}^{\infty}\exp(ak - ne^{ak}) + a\sum_{k=1}^{\infty}\frac{(-1)^k n^k k}{k!(e^{ak}-1)} - 2b\sum_{k=1}^{\infty}\mathrm{Im}\left(n^{-ikb}k\Gamma(ikb)\right) = 0 \quad (18)$$

2. Let $y(k) = \frac{(-1)^k \sin kv}{k}$ and $f(k) = k$, then we have $(M\Psi)(s) = \frac{\pi\csc(\pi s)\sin(\pi s)}{2s}$. Hence if $a, v > 0$, noting that

$$\sum_{k=1}^{\infty}\frac{(-1)^k\sin(kv)}{e^{ak}-1} = -\frac{1}{2}\sum_{k=1}^{\infty}\frac{\sin(v)}{\cos(v)+\cosh(ak)} \quad (19)$$

we have from the MSF.

$$\frac{a}{2}\tan\left(\frac{v}{2}\right) = v + 2a\sum_{k=1}^{\infty}\frac{(-1)^k\sin(kv)}{e^{ak}-1} + 2\pi\sum_{k=1}^{\infty}\mathrm{csch}\left(\frac{2k\pi^2}{a}\right)\sinh\left(\frac{2\pi kv}{a}\right) \quad (20)$$

It is known from tables that [6]

$$4\sum_{k=1}^{\infty}\frac{(-1)^k\sin(2kz)q^{2k}}{1-q^{2k}} = \tan(z) + \partial_z\left(\log(\theta_2(z,q))\right) \quad (21)$$

Thus we arrive at

$$2\pi\sum_{k=1}^{\infty}\frac{\sinh(2k\pi za)}{\sinh(k\pi^2 a)} = -2z - \frac{1}{a}\partial_z\left(\log(\theta_2(z,e^{-1/a}))\right) \quad (22)$$

3. From the relations [7]

$$\sum_{k=1}^{\infty}\frac{(-1)^k k}{e^{2\pi k/a}-1} = \frac{1}{8} - \frac{a}{4\pi} + \frac{a^2 K}{2\pi^2}(E-K) \quad (23)$$

$$\sum_{n=1}^{\infty}\frac{\cosh(2tn)}{n\sinh(\pi an)} = \log(P_0) - \log\left(\theta_4(it, e^{-a\pi})\right) \quad (24)$$

and

$$-\frac{1}{4} + \frac{a}{2\pi} + 2\sum_{k=1}^{\infty}\frac{(-1)^k k}{e^{2\pi k/a}-1} + 2a^2\sum_{k=1}^{\infty}\frac{k\cosh(ak\pi)}{\sinh(2ak\pi)} = 0 \quad (25)$$

from the Main Theorem we get

$$2\frac{\partial^2}{\partial t^2}\log\left(\theta_4\left(\frac{it\pi}{2}, e^{-2\pi a}\right)\right) = K(k_a)E(k_a) - K(k_a)^2 \quad (26)$$

whenever $\frac{K'(k_a)}{K(k_a)} = a$.

4. Also we have

$$\partial_x\left(e^{x^2 a/\pi}\frac{\theta_2(x, e^{-\pi/a})}{\theta_4(iax, e^{-a\pi})}\right) = 0 \quad (27)$$

5. If $a, v > 0$ and $L_s(x) = \sum_{k=1}^{\infty}\frac{x^k}{k^s}$, denotes the Polylogarithm function then

$$\frac{-v^3}{6} + av\log(2) + \frac{ia}{2}\left(L_2(-e^{-iv}) - L_2(-e^{iv})\right) = -2a\sum_{k=1}^{\infty}\frac{(-1)^k(\sin(kv)-kv)}{k^2(e^{ak}-1)} + \frac{a^2}{2\pi}\sum_{k=1}^{\infty}\mathrm{csch}\left(\frac{2k\pi^2}{a}\right)\frac{\sinh\left(\frac{2\pi kv}{a}\right) - \frac{2k\pi v}{a}}{k^2} \quad (28)$$

Proof

Let $y(k) = \frac{(-1)^k(\sin(kv)-kv)}{k^3}$ and $f(k) = k$, then if $\Psi(x) = \sum_{k=0}^{\infty}y(k)x^k$ we have $(M\Psi)(s) = \frac{\pi\csc(\pi s)(-sv+\sin(sv))}{s^3}$

In the same way we have

$$\frac{v^4}{24} - \frac{v^2 a\log(2)}{2} + \frac{a}{2}\left(L_3(-e^{-iv}) + L_3(-e^{iv})\right) + \frac{3a\zeta(3)}{4} = -2a\sum_{k=1}^{\infty}\frac{(-1)^k\left(\cos(kv) - \frac{k^2v^2}{2} - 1\right)}{(e^{ak}-1)k^3} +$$

$$+\frac{a^3}{4\pi^2}\sum_{k=1}^{\infty}\mathrm{csch}\left(\frac{2k\pi^2}{a}\right)\frac{1+\frac{2k^2\pi^2v^2}{a^2}-\cosh\left(\frac{2\pi kv}{a}\right)}{k^3} \quad (29)$$

where $\zeta(s)=\sum_{k=1}^{\infty}\frac{1}{k^s}$ is the Riemann Zeta function.

Or, if $ab=2\pi$ then

$$\frac{v^4}{24a}-\frac{v^2\log(2)}{2}+\frac{1}{2}\left(L_3(-e^{-iv})+L_3(-e^{iv})\right)+\frac{3}{4}\zeta(3)+$$
$$2\sum_{n=1}^{\infty}\frac{(-1)^n(\cos(nv)-1)}{n^3(e^{an}-1)}+v^2\sum_{n=1}^{\infty}\frac{(-1)^n}{n(e^{na}-1)}-$$
$$\frac{a^2}{4\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^3\sinh(bn\pi)}-\frac{v^2}{2}\sum_{n=1}^{\infty}\frac{1}{n\sinh(bn\pi)}+$$
$$\frac{a^2}{4\pi^2}\sum_{n=1}^{\infty}\frac{\cosh(bnv)}{n^3\sinh(bn\pi)}=0 \quad (30)$$

6. **A consequence of Jacobi's triple identity.** Note that the calculations of sums in Application 5 depend on finding the values

$$X(v,a)=C_1(a)v^2$$
$$=\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n(\cos(2nv)-1)}{n^3(e^{2na}-1)}-$$
$$\int_0^v\int_0^r\int_0^s\tan(t)dtdsdr-N(v,e^{-a}) \quad (31)$$

$$Y(v,b)=A_2+vB_2(b)+v^2C_2(b)=$$
$$=\frac{\pi}{4b^3}\sum_{n=1}^{\infty}\frac{\cosh(nvb)}{n^3\sinh(\pi nb)}+\frac{v^4}{192}+\pi bN\left(\frac{v}{2},e^{-\pi/b}\right)$$

where we have set

$$N(z,q):=\int_0^z\int_0^r\log\left(\theta_2(t,q)\right)dtdr \quad (32)$$

To find $C_1(a)$ we differentiate $X(v,a)$ twice with respect to v to get

$$X(v,a)=v^2C_1(a)=-v^2\sum_{n=1}^{\infty}\frac{(-1)^n\cos(2nv)}{n(e^{an}-1)}+$$
$$v^2\sum_{n=1}^{\infty}\frac{(-1)^n}{n(e^{na}-1)}-\frac{v^2}{2}\log(\theta_2(v,e^{-a})+\frac{v^2}{2}\log(\theta_2(e^{-a}))+$$
$$\frac{v^2}{2}\log(\cos(v))$$

where $\theta_j(0,q)=\theta_j(q)$.

Note use Theorem 1 with $f(t)=\left(\frac{\cos(ct)-1}{t}\right)$ along with Jacobi's triple identity [6] to arrive at

$$\sum_{n=1}^{\infty}\frac{(-1)^n\cos(2nv)}{n(e^{nb}-1)}=\frac{v^2}{b}-\frac{\log(2)}{2}+\frac{\log(1+e^{-2iv})}{4}+$$
$$\frac{\log(1+e^{2iv})}{4}-\frac{\log\left(\theta_4\frac{2i\pi v}{b},e^{-2\pi^2/b}\right)}{2}+$$
$$\frac{\log(f(-e^{-4\pi^2/b})}{2}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n(e^{bn}-1)}-$$
$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n\sinh\left(\frac{2n\pi^2}{b}\right)} \quad (33)$$

But it is known from tables that

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n(e^{nb}-1)}=-\frac{1}{24}\log\left(\frac{k_b}{16e^{-b}(1-k_b)^2}\right) \quad (34)$$

where k_b is the solution of $b=\frac{\pi K(1-k_b)}{K(k_b)}$, $K(x)=\int_0^{\pi/2}\frac{1}{\sqrt{1-x\sin^2(\theta)}}d\theta$ (note that when $b\in\mathbb{Q}_+^*$ then the k_b are algebraic numbers) and

$$\sum_{n=1}^{\infty}\frac{1}{n\sinh(an\pi)}=\log\left(f(-e^{-2a\pi})\right)-\log\left(\theta_4(e^{-a\pi})\right) \quad (35)$$

Hence we have

Proposition 2.1

$$-2\sum_{n=1}^{\infty}\frac{(-1)^n\sin^2(nv)}{n^3(e^{2an}-1)}=-\frac{v^4}{a}+v^2\log(2)-$$
$$\frac{v^2}{2}\log(1+e^{-2iv})-\frac{v^2}{2}\log(1+e^{2iv})+$$
$$v^2\log(\cos(v))-v^2\log\left(\theta_2(v,e^{-a})\right)-$$
$$v^2\log(\theta_4(e^{-\pi^2/a}))+v^2\log\left(\theta_4\left(\frac{i\pi v}{a},e^{-\pi^2/a}\right)\right)+$$
$$\frac{v^2}{12}\log\left(\frac{k_{2a}e^{2a}}{16(1-k_{2a})^2}\right)+2N(v,e^{-a})+$$
$$2\int_0^v\int_0^w\int_0^s\tan(t)dtdsdw \quad (36)$$

For Y we can find in the same way that

$$C_2(b) = \frac{v^2}{16} + \frac{\pi}{4b}\log\left(\theta_2\left(\frac{v}{2}, e^{-a}\right)\right) -$$

$$\frac{\pi}{4b}\log(f(-e^{-2\pi b})) \tag{37}$$

and $B_2(b) = 0$, where $f(-q) = \prod_{n=1}^{\infty}(1-q^n)$ is the q-product. Thus what remains is to find the value of $A_2(b)$. In this way we are led to

Proposition 2.2.

$$2\sum_{n=1}^{\infty}\frac{\sinh^2\left(\frac{nva}{2}\right)}{n^3\sinh(\pi na)} = \frac{5a^3v^4}{48\pi} +$$

$$\frac{a^2v^2}{2}\log\left(\theta_2\left(\frac{v}{2}, e^{-\pi/a}\right)\right) - \frac{a^2v^2}{2}\log\left(\theta_4\left(\frac{iav}{2}, e^{-\pi a}\right)\right)$$

$$+\frac{a^2v^2}{2}\log(f(-e^{-2a\pi})) - 4a^2N\left(\frac{v}{2}, e^{-\pi/a}\right) \tag{38}$$

References

1. C.K.Chui. 'An Introduction to Wavelets'. Academic Press. New York. 1992.
2. E.C. Tichmarsh. 'Introduction to the theory of Fourier integrals'. Oxford University Press, Amen House, London, 1948.
3. M.L. Glasser and N. Bagis, arXiv: 0812.0990.
4. N. Bagis and M.L. Glasser.' Integrals and Series Resulting from Two Sampling Theorems'. Journal of Sampling Theory in Signal and Image Processing. Vol. 5, No. 1, Jan. 2006, pp.89-97.
5. B.C. Berndt and J.L. Hafner. 'Two remarkable boubly exponential series transformations of Ramanujan'. J. Proceedings Mathematical Sciences Jan 2008, pp. 245-252.
6. E.T. Whittaker and G.N. Watson. 'A course on Modern Analysis'. Cambridge U.P. 1927
7. I.J. Zucker. 'The Summation of Series of Hyperbolic functions'. SIAM J. Math. Anal. 10. (1979). 192-206.

Received, January 10, 2010
In revised form, March 1, 2010