

On transformations involving generalized basic hypergeometric function of two variables*

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Abstract

In the present paper, transformations for basic analogue of the Fox's H-function of one and two variables have been derived by the application of the q-Leibniz rule for the product of two basic functions. Some special cases involving a basic analogue of Meijer's G-function are also derived.

Key words: Fractional q-derivative operator, q-Leibniz rule, basic analogue of Fox's H-function and basic Meijer's G-function.

Transformaciones de la función hipergeométrica básica generalizada de dos variables

Resumen

En el presente trabajo han sido derivadas transformaciones para la análoga básica de la función H de Fox de una y dos variables, aplicando la regla de Leibniz-q para el producto de dos funciones básicas. Algunos casos especiales que incluyen una análoga básica de la función G de Meijer son también derivados.

Palabras clave: Operador derivada-q fraccional, regla de Leibniz-q, análoga básica de la función H de Fox y la función G de Meijer básica.

1. Introduction

Recently in a couple of papers Yadav and Purohit [1, 2] have used the q-Leibniz rule for the fractional q-derivatives of the product of various basic hypergeometric function full stop. This has resulted in deduction of several transformations and expansion formulae involving the basic hypergeometric functions. Earlier Denis [3] and Shukla [4] have used the q-Leibniz rule to derive certain transformations for basic hypergeometric functions.

Recently Yadav et al. [5] have investigated the fractional q-calculus operators involving the

basic analogue of Fox's H-function and basic analogue of Meijer's G-function of two variables.

Motivated by the aforementioned work, we investigate the applications of the q-Leibniz rule to a product involving the basic analogue of Fox's H-function of two variables. This shall further be used to derive transformations involving the above mentioned functions.

The fractional q-differential operator of arbitrary order μ , cf. Al-Salam [6], is defined as:

$$D_{x,q}^{\mu} f(x) = \frac{1}{\Gamma_q(-\mu)} \int_0^x [x-yq]_{-\mu-1} f(y) d(y; q), \quad (1)$$

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where $\text{Re}(\mu) < 0$ and

$$[x - y]_v = x^v \prod_{n=0}^{\infty} \left[\frac{1 - (y/x)q^n}{1 - (y/x)q^{v+n}} \right], \tag{2}$$

where x and y are complex numbers with $x \neq 0$.

The basic integration cf. Gasper and Rahman [7] is defined as:

$$\int_0^x f(t) d(t; q) = x(1 - q) \sum_{k=0}^{\infty} q^k f(xq^k), \tag{3}$$

provided that the series converges.

By virtue of the equation (3), the equation (1) can be expressed as:

$$D_{x,q}^{\mu} f(x) = \frac{x^{-\mu}(1 - q)}{\Gamma_q(-\mu)} \sum_{k=0}^{\infty} q^k [1 - q^{k+1}]_{-\mu-1} f(xq^k), \tag{4}$$

where $\text{Re}(\mu) < 0$ and the q -gamma function cf. Gasper and Rahman [7], in various forms is given by

$$\Gamma_q(\alpha) = \frac{(q; q)_{\infty} (1 - q)^{1-\alpha}}{(q^{\alpha}; q)_{\infty}} = \frac{[1 - q]_{\alpha-1}}{(1 - q)^{\alpha-1}} = \frac{(q; q)_{\alpha-1}}{(1 - q)^{\alpha-1}}$$

$$0 < |q| < 1, \tag{5}$$

with $\alpha \neq 0, -1, -2, \dots$ and

$$(\alpha; q)_{\infty} = \prod_{j=0}^{\infty} (1 - \alpha q^j). \tag{6}$$

Further, for real or complex α and $0 < |q| < 1$, the q -shifted factorial is defined as:

$$(q^{\alpha}; q)_n = \begin{cases} 1, & \text{if } n = 0 \\ (1 - q^{\alpha})(1 - q^{\alpha+1}) \cdots (1 - q^{\alpha+n-1}), & \text{if } n \in N \tag{7} \\ [(1 - q^{\alpha-1})(1 - q^{\alpha-2}) \cdots (1 - q^{\alpha-n})]^{-1}, & \text{if } n \in Z^- \end{cases}$$

or

$$(\alpha; q)_n = \frac{(\alpha; q)_{\infty}}{(\alpha q^n; q)_{\infty}}. \tag{8}$$

In view of Agarwal [8], we have the q -extension of the Leibniz rule for the fractional q -derivatives for a product of two basic functions in terms

of a series involving the fractional q -derivatives of the function, in the following manner:

$$D_{x,q}^{\mu} \{U(x)V(x)\} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{-\frac{n(n+1)}{2}} (q^{-\mu}; q)_n}{(q; q)_n} \times D_{x,q}^{\mu-n} \{U(xq^n)\} D_{x,q}^n \{V(x)\}, \tag{9}$$

where $U(x)$ and $V(x)$ are two regular functions.

Following Saxena, Modi and Kalla [9], the basic analogue of the H-function of two variables is defined as:

$$H_{C,D,(P_1; Q_1),(P_2; Q_2)}^{A,(M_1; N_1),(M_2; N_2)} \left[\begin{matrix} z_1; q \\ z_2; q \end{matrix} \middle| \begin{matrix} (c; \gamma, \gamma') \\ (\alpha, \alpha); (\alpha', \alpha') \\ (d; \delta, \delta') \\ (b, \beta); (b', \beta') \end{matrix} \right] = \frac{1}{(2\pi i)^2} \times \int_{c_2^+} \int_{c_2^-} \frac{X_1(s; q) X_2(t; q) X_3(s, t; q) \pi^2 (z_1)^s (z_2)^t}{G(q^{1-s}) G(q^{1-t}) \sin \pi s \sin \pi t} ds dt, \tag{10}$$

where $0 < |q| < 1$ and

$$G(q^{\alpha}) = \left\{ \prod_{n=0}^{\infty} (1 - q^{\alpha+n}) \right\}^{-1} = \frac{1}{(q^{\alpha}; q)_{\infty}}. \tag{11}$$

Also

$$X_1(s, q) = \frac{\prod_{j=1}^{M_1} G(q^{b_j - \beta_j s}) \prod_{j=1}^{N_1} G(q^{1 - a_j + \alpha_j s})}{\prod_{j=M_1+1}^{Q_1} G(q^{1 - b_j + \beta_j s}) \prod_{j=N_1+1}^{P_1} G(q^{\alpha_j - \alpha_j s})}, \tag{12}$$

$$X_2(t, q) = \frac{\prod_{j=1}^{M_2} G(q^{b_j - \beta_j t}) \prod_{j=1}^{N_2} G(q^{1 - a_j + \alpha_j t})}{\prod_{j=M_2+1}^{Q_2} G(q^{1 - b_j + \beta_j t}) \prod_{j=N_2+1}^{P_2} G(q^{\alpha_j - \alpha_j t})}, \tag{13}$$

and

$$X_3(s, t; q) = \frac{\prod_{j=1}^A G(q^{1 - c_j + \gamma_j s + \gamma_j t})}{\prod_{j=A+1}^C G(q^{c_j - \gamma_j s - \gamma_j t}) \prod_{j=1}^D G(q^{1 - d_j + \delta_j s + \delta_j t})} \tag{14}$$

The coefficients γ_j and γ'_j , ($1 \leq j \leq C$); δ_j and δ'_j , ($1 \leq j \leq D$); α_j ($1 \leq j \leq P_1$), α'_j ($1 \leq j \leq P_2$), β_j ($1 \leq j \leq Q_1$), β'_j ($1 \leq j \leq Q_2$) are positive numbers, $A, C, D, P_1, P_2, Q_1, Q_2, M_1, M_2, N_1$ and N_2 are non negative integers, satisfying the inequalities $0 \leq A \leq C, 0 \leq M_i \leq Q_i, 0 \leq N_i \leq P_i, D > 0; \forall i \in \{1,2\}$.

The contours C_1^* and C_2^* are lines parallel to $\text{Re}(w_1s) = 0$ ($i = 1,2$), with indentations, if necessary, in such a manner that all the poles of $G(q^{b_j - \beta_j s})$ for $j \in \{1, \dots, M_1\}$ and $G(q^{b'_j - \beta'_j t})$ for $j \in \{1, \dots, M_2\}$, lie to the right and those of $G(q^{1-c_j + \gamma_j s + \gamma'_j t})$ for $j \in \{1, \dots, A\}$, $G(q^{1-a_j + \alpha_j s})$ for $j \in \{1, \dots, N_1\}$ and $G(q^{1-a'_j + \alpha'_j s})$ for $j \in \{1, \dots, N_2\}$ lie to the left of the contours. An empty product is interpreted as unity. The poles of the integrand are assumed to be simple. The integral converges if $\text{Re}[s \log(z_1) - \log \sin \pi s] < 0$ and $\text{Re}[t \log(z_2) - \log \sin \pi t] < 0$ for large values of $|s|$ and $|t|$ on the contours i.e. if $|\{\arg(z_i) - w_2 w_1^{-1} \log|z_i|\}| < \pi$ for $i = 1,2$. Where $0 < |q| < 1$ such that $\log q = -w = -(w_1 + iw_2)$, w_1 and w_2 being real numbers.

If we set $\alpha = \alpha' = \beta = \beta' = \gamma = \gamma' = \delta = \delta' = 1$ then the definition (10) reduces to a basic analogue of Meijer's G-function of two variables as under:

$$H_{C,D,(P_1:Q_1),(P_2:Q_2)}^{A,(M_1:N_1),(M_2:N_2)} \left[\begin{matrix} z_1 \\ z_2 \end{matrix}; q \left| \begin{matrix} (c;1,1) \\ (a,1); (a',1) \\ (d;1,1) \\ (b,1); (b',1) \end{matrix} \right. \right] =$$

$$G_{C,D,(P_1:Q_1),(P_2:Q_2)}^{A,(M_1:N_1),(M_2:N_2)} \left[\begin{matrix} z_1 \\ z_2 \end{matrix}; q \left| \begin{matrix} c_1, \dots, c_C \\ a, \dots, a_{P_1}; a'_1, \dots, a'_{P_2} \\ d_1, \dots, d_D \\ b_1, \dots, b_{Q_1}; b'_1, \dots, b'_{Q_2} \end{matrix} \right. \right] =$$

$$\frac{1}{(2\pi i)^2} \int_{C_2^*} \int_{C_1^*} \frac{Y_1(s; q) Y_2(t; q) Y_3(s, t; q) \pi^2 (z_1)^s (z_2)^t}{G(q^{1-s}) G(q^{1-t}) \sin \pi s \sin \pi t} ds dt \tag{15}$$

where

$$Y_1(s, q) = \frac{\prod_{j=1}^{M_1} G(q^{b_j - s}) \prod_{j=1}^{N_1} G(q^{1-a_j + s})}{\prod_{j=M_1+1}^{Q_1} G(q^{1-b_j + s}) \prod_{j=N_1+1}^{P_1} G(q^{a_j - s})} \tag{16}$$

$$Y_2(t; q) = \frac{\prod_{j=1}^{M_2} G(q^{b'_j - t}) \prod_{j=1}^{N_2} G(q^{1-a'_j + t})}{\prod_{j=M_2+1}^{Q_2} G(q^{1-b'_j + t}) \prod_{j=N_2+1}^{P_2} G(q^{a'_j - t})} \tag{17}$$

and

$$Y_3(s, t; q) = \frac{\prod_{j=1}^A G(q^{1-c_j + s+t})}{\prod_{j=A+1}^C G(q^{c_j - s-t}) \prod_{j=1}^D G(q^{1-d_j + s+t})} \tag{18}$$

Further, we observe that for $A = C = D = 0$ the basic analogue of Fox's H-function of two variables given by (10) reduces to a product of two basic Fox's H-functions of one variables cf. Saxena, Modi and Kalla [9], as under:

$$H_{0,0,(P_1:Q_1),(P_2:Q_2)}^{0,(M_1:N_1),(M_2:N_2)} \left[\begin{matrix} z_1 \\ z_2 \end{matrix}; q \left| \begin{matrix} (a, \alpha); (a', \alpha') \\ (b, \beta); (b', \beta') \end{matrix} \right. \right] =$$

$$H_{P_1, Q_1}^{M_1, N_1} \left[z_1; q \left| \begin{matrix} (a, \alpha) \\ (b, \beta) \end{matrix} \right. \right] H_{P_2, Q_2}^{M_2, N_2} \left[z_2; q \left| \begin{matrix} (a', \alpha') \\ (b', \beta') \end{matrix} \right. \right], \tag{19}$$

where the basic analogue of Fox's H-function of one variable due to Saxena, Modi and Kalla [10], is given by

$$H_{P_1, Q_1}^{M_1, N_1} \left[x; q \left| \begin{matrix} (a, \alpha) \\ (b, \beta) \end{matrix} \right. \right] = \frac{1}{2\pi i} \times$$

$$\int_C \frac{\prod_{j=1}^{M_1} G(q^{b_j - \beta_j s}) \prod_{j=1}^{N_1} G(q^{1-a_j + \alpha_j s}) \pi x^s}{\prod_{j=M_1+1}^{Q_1} G(q^{1-b_j + \beta_j s}) \prod_{j=N_1+1}^{P_1} G(q^{a_j - \alpha_j s}) G(q^{1-s}) \sin \pi s} ds, \tag{20}$$

where $0 \leq M_1 \leq Q_1, 0 \leq N_1 \leq P_1$, α_j 's and β_j 's all positive integers. The contour C is a line parallel to $\text{Re}(ws) = 0$ with indentations if necessary, in such a manner that all the poles of $G(q^{b_j - \beta_j s})$, $1 \leq j \leq M_1$ are to the right, and those of $G(q^{1-a_j + \alpha_j s})$, $1 \leq j \leq N_1$ to the left of C . The integral converges if $\text{Re}[s \log(x) - \log \sin \pi s] < 0$ for large values of $|s|$ on the contour C i.e. if $|\{\arg(x) - w_2 w_1^{-1} \log|x|\}| < \pi$, where $0 < |q| < 1$,

$\log q = -w = -(w_1 + iw_2)$, w, w_1, w_2 are definite quantities, w_1 and w_2 being real.

Further, if we set $\alpha_i = \beta_j = 1, \forall i$ and j in the equation (20), we obtain the basic analogue of Meijer's G-function, due to Saxena, Modi and Kalla [10], namely

$$H_{P_1, Q_1}^{M_1, N_1} \left[x; q \left| \begin{matrix} (a, 1) \\ (b, 1) \end{matrix} \right. \right] \equiv G_{P_1, Q_1}^{M_1, N_1} \left[x; q \left| \begin{matrix} a_1, \dots, a_{P_1} \\ b_1, \dots, b_{Q_1} \end{matrix} \right. \right] = \frac{1}{2\pi i} \times \int_C \frac{\prod_{j=1}^{M_1} G(q^{b_j-s}) \prod_{j=1}^{N_1} G(q^{1-a_j+s}) \pi x^s}{\prod_{j=M_1+1}^{Q_1} G(q^{1-b_j+s}) \prod_{j=N_1+1}^{P_1} G(q^{a_j-s}) G(q^{1-s}) \sin \pi s} ds, \tag{21}$$

where $0 \leq M_1 \leq Q_1, 0 \leq N_1 \leq P_1$ and $\text{Re}[s \log(x) - \log \sin \pi s] < 0$.

2. Transformations involving a basic analogue of Fox's H-function of two variables

In this section, we shall establish certain theorems involving some transformations associated with the basic analogue of the Fox's H-function and Meijer's G-function of two variables.

If $\text{Re}(\mu) < 0, \text{Re}[s \log(z_1) - \log \sin \pi s] < 0, \text{Re}[t \log(z_2) - \log \sin \pi t] < 0, \rho$ and σ being any positive integers, then for $\lambda \neq 0, -1, -2, \dots$, the following theorems holds:

Theorem 1

$$H_{C+1, D+1, (P_1; Q_1), (P_2; Q_2)}^{A+1, (M_1; N_1), (M_2; N_2)} \left[\begin{matrix} z_1 x^\rho \\ z_2 x^\sigma \end{matrix}; q \left| \begin{matrix} (1-\lambda; \rho, \sigma)(c; \gamma, \gamma') \\ (a, \alpha); (a', \alpha') \\ (d; \delta, \delta')(1-\lambda+\mu; \rho, \sigma) \\ (b, \beta); (b', \beta') \end{matrix} \right. \right] = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n\lambda + n(n-1)}{2}} (q^{-\mu}; q)_n}{(q; q)_n (q^\lambda; q)_{n-\mu}} \times H_{C+1, D+1, (P_1; Q_1), (P_2; Q_2)}^{A+1, (M_1; N_1), (M_2; N_2)} \left[\begin{matrix} z_1 x^\rho \\ z_2 x^\sigma \end{matrix}; q \left| \begin{matrix} (0; \rho, \sigma)(c; \gamma, \gamma') \\ (a, \alpha); (a', \alpha') \\ (d; \delta, \delta')(n; \rho, \sigma) \\ (b, \beta); (b', \beta') \end{matrix} \right. \right] \tag{22}$$

Theorem 2

$$H_{1,1, (P_1; Q_1), (P_2; Q_2)}^{1, (M_1; N_1), (M_2; N_2)} \left[\begin{matrix} z_1 x^\rho \\ z_2 x^\sigma \end{matrix}; q \left| \begin{matrix} (1-\lambda; \rho, \sigma) \\ (a, \alpha); (a', \alpha') \\ (1-\lambda+\mu; \rho, \sigma) \\ (b, \beta); (b', \beta') \end{matrix} \right. \right] = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n\lambda + n(n-1)}{2}} (q^{-\mu}; q)_n}{(q; q)_n} \times H_{P_1+1, Q_1+1}^{M_1, N_1+1} \left[z_1 (xq^n)^\rho; q \left| \begin{matrix} (1-\lambda; \rho), (a, \alpha) \\ (b, \beta), (1-\lambda+\mu-n, \rho) \end{matrix} \right. \right] \times H_{P_2+1, Q_2+1}^{M_2, N_2+1} \left[z_2 x^\sigma; q \left| \begin{matrix} (0, \sigma), (a', \alpha') \\ (b', \beta'), (n, \sigma) \end{matrix} \right. \right], \tag{23}$$

Theorem 3

$$H_{C, D, (P_1+1; Q_1+1), (P_2; Q_2)}^{A, (M_1, N_1+1), (M_2; N_2)} \left[\begin{matrix} z_1 x^\rho \\ z_2 \end{matrix}; q \left| \begin{matrix} (c; \gamma, \gamma') \\ (1-\lambda, \rho), (a, \alpha); (a', \alpha') \\ (d; \delta, \delta') \\ (b, \beta), (1-\lambda+\mu; \rho); (b', \beta') \end{matrix} \right. \right] = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n\lambda + n(n-1)}{2}} (q^{-\mu}; q)_n}{(q; q)_n (q^\lambda; q)_{n-\mu}} \times H_{C, D, (P_1+1; Q_1+1), (P_2; Q_2)}^{A, (M_1, N_1+1), (M_2; N_2)} \left[\begin{matrix} z_1 x^\rho \\ z_2 \end{matrix}; q \left| \begin{matrix} (c; \gamma, \gamma') \\ (0, \rho), (a, \alpha); (a', \alpha') \\ (d; \delta, \delta') \\ (b, \beta), (n; \rho); (b', \beta') \end{matrix} \right. \right]. \tag{24}$$

Proof of (22): On taking, in the q-Leibniz rule (9) $U(x) = x^{\lambda-1}$ and

$$V(x) = H_{C, D, (P_1; Q_1), (P_2; Q_2)}^{A, (M_1; N_1), (M_2; N_2)} \left[\begin{matrix} z_1 x^\rho \\ z_2 x^\sigma \end{matrix}; q \left| \begin{matrix} (c; \gamma, \gamma') \\ (a, \alpha); (a', \alpha') \\ (d; \delta, \delta') \\ (b, \beta); (b', \beta') \end{matrix} \right. \right]$$

we get,

$$D_{x,q}^\mu \left\{ x^{\lambda-1} H_{C, D, (P_1; Q_1), (P_2; Q_2)}^{A, (M_1; N_1), (M_2; N_2)} \left[\begin{matrix} z_1 x^\rho \\ z_2 x^\sigma \end{matrix}; q \left| \begin{matrix} (c; \gamma, \gamma') \\ (a, \alpha); (a', \alpha') \\ (d; \delta, \delta') \\ (b, \beta); (b', \beta') \end{matrix} \right. \right] \right\} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n(n+1)}{2}} (q^{-\mu}; q)_n}{(q; q)_n} D_{x,q}^{\mu-n} \{ (xq^n)^{\lambda-1} \} \times$$

$$D_{x,q}^n \left\{ H_{C,D,(P_1;\mathcal{Q}_1),(P_2;\mathcal{Q}_2)}^{A,(M_1:N_1),(M_2:N_2)} \left[\begin{matrix} z_1 x^\rho \\ z_2 x^\sigma \end{matrix} ; q \left| \begin{matrix} (c; \gamma, \gamma') \\ (a, \alpha); (a', \alpha') \\ (d; \delta, \delta') \\ (b, \beta); (b', \beta') \end{matrix} \right. \right] \right\}.$$

Following the recent communication of authors [5] we have:

$$D_{x,q}^\mu \left\{ x^{\lambda-1} H_{C,D,(P_1;\mathcal{Q}_1),(P_2;\mathcal{Q}_2)}^{A,(M_1:N_1),(M_2:N_2)} \left[\begin{matrix} z_1 x^\rho \\ z_2 x^\sigma \end{matrix} ; q \left| \begin{matrix} (c; \gamma, \gamma') \\ (a, \alpha); (a', \alpha') \\ (d; \delta, \delta') \\ (b, \beta); (b', \beta') \end{matrix} \right. \right] \right\} =$$

$$(1-q)^{-\mu} x^{\lambda-\mu-1} H_{C+1,D+1,(P_1;\mathcal{Q}_1),(P_2;\mathcal{Q}_2)}^{A+1,(M_1:N_1),(M_2:N_2)} \times$$

$$\left[\begin{matrix} z_1 x^\rho \\ z_2 x^\sigma \end{matrix} ; q \left| \begin{matrix} (1-\lambda; \rho, \sigma)(c; \gamma, \gamma') \\ (a, \alpha); (a', \alpha') \\ (d; \delta, \delta')(1-\lambda+\mu; \rho, \sigma) \\ (b, \beta); (b', \beta') \end{matrix} \right. \right], \tag{25}$$

where $\text{Re}[s \log(z_1) - \log \sin \pi s] < 0$ and $\text{Re}[t \log(z_2) - \log \sin \pi t] < 0$.

If we set $\lambda = 1$ in the above result (25), we obtain the following transformation involving Fox's H-function of two variables:

$$D_{x,q}^\mu \left\{ H_{C,D,(P_1;\mathcal{Q}_1),(P_2;\mathcal{Q}_2)}^{A,(M_1:N_1),(M_2:N_2)} \left[\begin{matrix} z_1 x^\rho \\ z_2 x^\sigma \end{matrix} ; q \left| \begin{matrix} (c; \gamma, \gamma') \\ (a, \alpha); (a', \alpha') \\ (d; \delta, \delta') \\ (b, \beta); (b', \beta') \end{matrix} \right. \right] \right\} =$$

$$(1-q)^{-\mu} x^{-\mu} H_{C+1,D+1,(P_1;\mathcal{Q}_1),(P_2;\mathcal{Q}_2)}^{A+1,(M_1:N_1),(M_2:N_2)} \times$$

$$\left[\begin{matrix} z_1 x^\rho \\ z_2 x^\sigma \end{matrix} ; q \left| \begin{matrix} (0; \rho, \sigma)(c; \gamma, \gamma') \\ (a, \alpha); (a', \alpha') \\ (d; \delta, \delta')(\mu; \rho, \sigma) \\ (b, \beta); (b', \beta') \end{matrix} \right. \right], \tag{26}$$

where $\text{Re}[s \log(z_1) - \log \sin \pi s] < 0$ and $\text{Re}[t \log(z_2) - \log \sin \pi t] < 0$.

On using the relation (26) and a fundamental result of fractional q-calculus, namely

$$D_{x,q}^\mu (x^{\lambda-1}) = \frac{\Gamma_q(\lambda)}{\Gamma_q(\lambda-\mu)} x^{\lambda-\mu-1}, \quad (\lambda \neq 0, -1, -2, \dots), \tag{27}$$

we arrive at the Theorem 1 after some simplifications.

Proof of (23): On setting

$$U(x) = x^{\lambda-1} H_{P_1,\mathcal{Q}_1}^{M_1,N_1} \left[z_1 x^\rho ; q \left| \begin{matrix} (a, \alpha) \\ (b, \beta) \end{matrix} \right. \right]$$

and,

$$V(x) = H_{P_2,\mathcal{Q}_2}^{M_2,N_2} \left[z_2 x^\sigma ; q \left| \begin{matrix} (a', \alpha') \\ (b', \beta') \end{matrix} \right. \right],$$

in the q-Leibniz rule (9), we obtain

$$D_{x,q}^\mu \left\{ x^{\lambda-1} H_{P_1,\mathcal{Q}_1}^{M_1,N_1} \left[z_1 x^\rho ; q \left| \begin{matrix} (a, \alpha) \\ (b, \beta) \end{matrix} \right. \right] H_{P_2,\mathcal{Q}_2}^{M_2,N_2} \left[z_2 x^\sigma ; q \left| \begin{matrix} (a', \alpha') \\ (b', \beta') \end{matrix} \right. \right] \right\} =$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{-\frac{n(n+1)}{2}} (q^{-\mu}; q)_n}{(q; q)_n} \times$$

$$D_{x,q}^{\mu-n} \left\{ (xq^n)^{\lambda-1} H_{P_1,\mathcal{Q}_1}^{M_1,N_1} \left[z_1 (xq^n)^\rho ; q \left| \begin{matrix} (a, \alpha) \\ (b, \beta) \end{matrix} \right. \right] \right\} \times$$

$$D_{x,q}^\mu \left\{ H_{P_2,\mathcal{Q}_2}^{M_2,N_2} \left[z_2 x^\sigma ; q \left| \begin{matrix} (a', \alpha') \\ (b', \beta') \end{matrix} \right. \right] \right\}. \tag{28}$$

In view of the known results due to Yadav, Purohit and Vyas [5], namely

$$D_{x,q}^\mu \left\{ x^{\lambda-1} H_{P_1,\mathcal{Q}_1}^{M_1,N_1} \left[z_1 x^\rho ; q \left| \begin{matrix} (a, \alpha) \\ (b, \beta) \end{matrix} \right. \right] H_{P_2,\mathcal{Q}_2}^{M_2,N_2} \left[z_2 x^\sigma ; q \left| \begin{matrix} (a', \alpha') \\ (b', \beta') \end{matrix} \right. \right] \right\} =$$

$$(1-q)^{-\mu} x^{\lambda-\mu-1} H_{1,1,(P_1;\mathcal{Q}_1),(P_2;\mathcal{Q}_2)}^{1,(M_1:N_1),(M_2:N_2)} \times$$

$$\left[\begin{matrix} z_1 x^\rho \\ z_2 x^\sigma \end{matrix} ; q \left| \begin{matrix} (1-\lambda; \rho, \sigma) \\ (a, \alpha); (a', \alpha') \\ (1-\lambda+\mu; \rho, \sigma) \\ (b, \beta); (b', \beta') \end{matrix} \right. \right], \tag{29}$$

$$D_{x,q}^\mu \left\{ x^{\lambda-1} H_{C,D,(P_1;\mathcal{Q}_1),(P_2;\mathcal{Q}_2)}^{A,(M_1:N_1),(M_2:N_2)} \left[\begin{matrix} z_1 x^\rho \\ z_2 \end{matrix} ; q \left| \begin{matrix} (c; \gamma, \gamma') \\ (a, \alpha); (a', \alpha') \\ (d; \delta, \delta') \\ (b, \beta); (b', \beta') \end{matrix} \right. \right] \right\} =$$

$$(1-q)^{-\mu} x^{\lambda-\mu-1} H_{C,D,(P_1;\mathcal{Q}_1),(P_2;\mathcal{Q}_2)}^{A,(M_1:N_1),(M_2:N_2)} \times$$

$$\left[\begin{matrix} z_1 x^\rho \\ z_2 \end{matrix} ; q \left| \begin{matrix} (c; \gamma, \gamma') \\ (1-\lambda, \rho), (a, \alpha); (a', \alpha') \\ (d; \delta, \delta') \\ (b, \beta), (1-\lambda+\mu, \rho); (b', \beta') \end{matrix} \right. \right], \tag{30}$$

by assigning $A = C = D = 0$, in the above equations (29), (30) and using the result (19) we obtain

the following fractional q-derivative formulae involving Fox's H-function:

$$D_{x,q}^{\mu-n} \left\{ (xq^n)^{\lambda-1} H_{P_1, Q_1}^{M_1, N_1} \left[z_1 (xq^n)^\rho; q \left| \begin{matrix} (a, \alpha) \\ (b, \beta) \end{matrix} \right. \right] \right\} = \frac{x^{n-\mu+\lambda-1} q^{n(\lambda-1)}}{(1-q)^{\mu-n}} H_{P_1+1, Q_1+1}^{M_1, N_1+1} \times \left[z_1 (xq^n)^\rho; q \left| \begin{matrix} (1-\lambda, \rho)(a, \alpha) \\ (b, \beta), (1-\lambda+\mu-n, \rho) \end{matrix} \right. \right], \quad (31)$$

and

$$D_{x,q}^\mu \left\{ H_{P_2, Q_2}^{M_2, N_2} \left[z_2 x^\sigma; q \left| \begin{matrix} (\alpha', \alpha') \\ (b', \beta') \end{matrix} \right. \right] \right\} = \frac{x^{-n}}{(1-q)^n} H_{P_2+1, Q_2+1}^{M_2, N_2+1} \left[z_2 x^\sigma; q \left| \begin{matrix} (0, \sigma), (\alpha', \alpha') \\ (b', \beta'), (n, \sigma) \end{matrix} \right. \right]. \quad (32)$$

On substituting (29), (31) and (32) in (28) we arrive at the Theorem 2 after some simplifications.

Proof of (24): The proof of Theorem 3 is similar to that of the Theorem 1; for sake of brevity we omit the proof.

3. Applications

The q-extension of the H-function of two variables defined by (10) in terms of the Mellin-Barnes type of basic contour integrals, possess the advantage that a number of q-special functions (including Fox's H-function of one variable) happen to be the particular cases of this function. The transformations deduced in the previous section can find many applications giving rise to the transformations for various q-special functions, which are special cases of the Fox's H-function.

For example, if we set $\alpha = \alpha' = \beta = \beta' = \gamma = \gamma' = \delta = \delta' = \rho = \sigma = 1$ in the Theorem 1 and Theorem 2, we respectively obtain the following results involving Meijer's G-function of two variables:

$$(i) \quad G_{C+1, D+1, (P_1: Q_1), (P_2: Q_2)}^{A+1, (M_1: N_1), (M_2: N_2)} \left[z_1 x; q \left| \begin{matrix} 1-\lambda; c_1, \dots, c_C \\ a_1, \dots, a_{P_1}; a'_1, \dots, a'_{P_2} \\ d_1, \dots, d_D; 1-\lambda+\mu \\ b_1, \dots, b_{Q_1}; b'_1, \dots, b'_{Q_2} \end{matrix} \right. \right] =$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n\lambda+n(n-1)}{2}} (q^{-\mu}; q)_n}{(q; q)_n (q^\lambda; q)_{n-\mu}} G_{C+1, D+1, (P_1: Q_1), (P_2: Q_2)}^{A+1, (M_1: N_1), (M_2: N_2)} \times \left[z_1 x; q \left| \begin{matrix} 0; c_1, \dots, c_C \\ a_1, \dots, a_{P_1}; a'_1, \dots, a'_{P_2} \\ d_1, \dots, d_D; n \\ b_1, \dots, b_{Q_1}; b'_1, \dots, b'_{Q_2} \end{matrix} \right. \right]. \quad (33)$$

$$(ii) \quad G_{1, 1, (P_1: Q_1), (P_2: Q_2)}^{1, (M_1: N_1), (M_2: N_2)} \left[z_1 x; q \left| \begin{matrix} 1-\lambda \\ a_1, \dots, a_{P_1}; a'_1, \dots, a'_{P_2} \\ 1-\lambda+\mu \\ b_1, \dots, b_{Q_1}; b'_1, \dots, b'_{Q_2} \end{matrix} \right. \right] = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n\lambda+n(n-1)}{2}} (q^{-\mu}; q)_n}{(q; q)_n} G_{P_1+1, Q_1+1}^{M_1, N_1+1} \left[z_1 (xq^n); q \left| \begin{matrix} 1-\lambda, a_1, \dots, a_{P_1} \\ b_1, \dots, b_{Q_1}, 1-\lambda+\mu-n \end{matrix} \right. \right] \times G_{P_2+1, Q_2+1}^{M_2, N_2+1} \left[z_2 x; q \left| \begin{matrix} 0, a'_1, \dots, a'_{P_2} \\ b'_1, \dots, b'_{Q_2}, n \end{matrix} \right. \right]. \quad (34)$$

On putting $A = C = D = 0, \rho = 1$ and $z_1 = 1$ in the Theorem 3, we obtain a known result due to Yadav and Purohit [2, p.323, eq. (2.1)].

We conclude with an observation that the method used here can be employed to yield a variety of interesting results involving the expansions and transformations for the generalized basic hypergeometric functions of two variables.

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