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On totally contact umbilical submanifolds of a manifold with a sasakian 3-structure

Constantin Călin

Technical University "Gh.Asachi" Iaşi. Department of Mathematics 6600 Iaşi Romania.
E-mail adress:calin@math.tuiasi.ro

Abstract

In our paper [5] we proved that any totally contact umbilical submanifold M of a manifold with a Sasakian 3-structure with dim $\mu_x^{\perp} > 1$, $\forall x \in M$, is totally contact geodesic. In the present paper we solve the remaining cases. Namely, when dim $\mu_x^{\perp} = 0$, or dim $\mu_x^{\perp} = 1$, M is totally contact geodesic or an intrinsic sphere respectively.

Key words: Sasakian-3 structure; totally contact umbilical; totally contact geodesic; extrinsic sphere.

Sobre subvariedades con contacto umbilical completo de una variedad con una estructura-3 sasakian

Resumen

En nuestro trabajo [5] probamos que cualquier subvariedad con contacto umbilical completo de una variedad con una estructura-3 Sasakian con dim $\mu_x^{\perp} > 1$, para todo x que pertenece a M, es de contacto geodésico total. En el presente trabajo resolvemos los casos restantes. A saber, cuando dim $\mu_x^{\perp} = 0$ ó dim $\mu_x^{\perp} = 1$, *M* es de contacto geodésico total o una esfera intrinseca, rectivamente.

Palabras clave: Estructura 3-Sasakian, contacto umbilical completo, contacto geodésico total, esfera extrinseca.

Introduction

The notion of CR-submanifold has been introduced by A. Bejancu [1] for the Kaehler manifolds, by A. Bejancu-N. Papaghiuc [2] for the Sasakian manifolds (called semi-invariant submanifolds) and by M. Barros- B.Y. Chen-F. Urbano [3] for the quaternionic manifolds. Later, CR-submanifolds have been intensively studied from different points of view, several important results have been obtained, some of them being brought together in [1]. Also some important results have been obtained in [4] about QR-submanifolds of quaternionic Kaehlerian manifolds and in [2] on semi-invariant submanifolds of a manifold with a Sasakian 3-structure.

It is well known (see [5]) that the tangent bundle *TM* of a semi-invariant submanifold *M* (called also contact CR-submanifolds), tangent to the structure vector field ξ, has the decomposition $TM = D \oplus D^{\perp} \oplus \{\xi\}$, where D and $D^{\perp}$ are the invariant and anti-invariant distributions on *M*, with respect to the structure tensor field f on manifold $\widetilde{M}$. Equivalently, M is a semi-invariant submanifold of a manifold $\widetilde{M}$ if its normal bundle $TM^{\perp}$ has the decomposition $TM^{\perp} = \mu \oplus \mu^{\perp}$, where μ and $\mu^{\perp}$ are invariant and anti-invariant subbundles of $TM^{\perp}$ with respect to f. The equivalence fails in the case of manifold with a Sasakian 3-structure.In this case the distribution $D^{\perp}$ is not anti-invariant to the structure tensor field.

According to a known result (see [2]) a totally contact umbilical semi-invariant submanifold of a manifold with a Sasakian 3-structure with $\dim \mu \frac{1}{x} > 1$, for any $x \in M$, is totally contact geodesic. The main purpose of the present paper is to study the remaining cases. More precisely we prove that *M* is totally contact geodesic submanifold of $\tilde{M}$, if $\dim \mu_x^{\perp} = 0$. If $\dim \mu_x^{\perp} = 1$, $x \in M$, but *M* is not totally contact geodesic, then *M* is extrinsic sphere.

Preliminaries

Let $\tilde{M}$ be a (4n+3)-dimensional differentiable manifold with an almost contact metric 3-structure (f_a, ξ_a, η_a, g), $a \in \{1,2,3\}$. Then we have

a) $f_a^2 = -I + \eta_a \otimes \xi_a$, (b) $\eta_a(\xi_b) = \delta_{ab}$

(c) $f_a(\xi_b) = -f_b(\xi_a) = \xi_c$, (d) $\eta_a \circ f_b = -\eta_b \circ f_a = \eta_c$,

(e) $f_a \circ f_b - \eta_b \otimes \xi_a = -f_b \circ f_a + \eta_a \otimes \xi_b = f_c$,

(f) $\eta_a(X) = g(X, \xi_a)$

(g) $g(f_a X, f_a Y) = g(X,Y) - \eta_a(X)\eta_a(Y)$

for any cyclic permutation (a, b, c) of (1, 2, 3), where *X* and *Y* are the vector fields tangent to $\tilde{M}$, δ is the Kronecker's delta. Then $\tilde{M}$ is called a manifold with a Sasakian 3-structure, if each (f_a, ξ_a, η_a, g) is a Sasakian 3-structure, i.e. (see [6]):

a) $(\tilde{\nabla}_X f_a) Y = g(X,Y)\xi_a - \eta_a(Y)X$,

b) $\tilde{\nabla}_X \xi_a = -f_a X$, $a \in \{1,2,3\}$ (1.2)

for any vector fields X, Y tangent to $\tilde{M}$ where $\tilde{\nabla}$ is the Levi-Civita connection on $\tilde{M}$. It is easy to see that $[\xi_a, \xi_b] = 2\xi_c$ for any cyclic permutation (a, b, c) of (1, 2, 3). Throughout the paper, all manifolds and maps are supposed differentiable of class C^{∞}. We denote by *F(M)* the module of the differentiable functions on $\tilde{M}$ and by $\Gamma(E)$ the module of smooth sections of a vector bundle E over $\tilde{M}$. We use the same notations for any manifolds involved in the study.

The curvature tensor K of $\tilde{M}$ is defined by

$$K(X,Y)Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]} Z,$$
$$\forall X, Y, Z \in \Gamma(T\tilde{M}).$$

Because the structure tensor field f_a verifies (1.2a) then the curvature tensor field K verify

a) $K(X,Y)f_a Z = f_a K(X,Y)Z + g(f_a X, Z)Y - g(Y,Z) f_a X + g(X,Z) f_a Y - g(f_a Y, Z)X$

b) $g(K(f_a X, f_a Y) f_a Z, f_a W) = g(K(X,Y)Z, W) - \eta_a(Y)\eta_a(Z) g(X,W) - \eta_a(X)\eta_a(W) g(Y,Z) + \eta_a(X)\eta_a(Z) g(Y,W) + \eta_a(Y)\eta_a(W) g(X,Z)$,

c) $K(X, \xi_a)Y = \eta_a(Y)X - g(X,Y)\xi_a$, $a \in \{1,2,3\}$, $\forall X, Y, Z, W \in \Gamma(T\tilde{M})$ (1.3)

Now, let M be a m-dimensional Riemannian manifold isometrically immersed in $\tilde{M}$, and suppose that the structure vector fields ξ_1, ξ_2, ξ_3 of $\tilde{M}$ are tangent to M. We denote by TM and $TM^{\perp}$ the tangent bundle and the normal bundle to M, repectively. We also denote by $\{\xi\}$ the distribution spanned by ξ_1, ξ_2, ξ_3 on M. The induced metric tensor on M will be denoted by the same symbol g.

The submanifold M of a manifold with a Sasakian 3-structure is called semi-invariant submanifold (see [2]) if there exists a vector subbundle μ of $TM^{\perp}$ such that

$$f_a(\mu) = \mu;\ f_a(\mu^{\perp}) \subseteq TM,\ a \in \{1,2,3\},$$

where $\mu^{\perp}$ is the complementary orthogonal bundle to μ in $TM^{\perp}$. It is easy to see that any real hypersurface of $\tilde{M}$ is a semi-invariant submanifold. Next, denote $f_a(\mu_x^{\perp})$ by D_{ax}, $a \in \{1,2,3\}$ $x \in M$. By using (1.1e) and (1.1g) it is obtained that D_{1x}, D_{2x}, D_{3x} are mutually orthogonal subspaces of xxx and have the same dimension s as the dimension of $T_x M$. We note that the subspaces D_{ax}, $a \in \{1,2,3\}$ do not define in general a distribution on *M*, but the maping.

$$D^{\perp} : x \rightarrow D_x^{\perp} = D_{1x} \otimes D_{2x} \otimes D_{3x},$$

is a 3s-dimensional distribution on M ($s = \dim \mu_x^{\perp}$). By straightforward calculation we deduce

$$a)\ f_a(D_{ax}) = \mu_x^{\perp};\ b)\ f_a(D_{bx}) = D_{cx} \qquad (1.4)$$

for each $x \in M$, where (a, b, c) is a cyclic permutation of (1,2,3). We denote by D the complementary orthogonal distribution to $D^{\perp} \otimes \{\xi\}$ in TM. It follows that the distribution D is invariant with respect to the action of f_1, f_2, f_3, that is $f_a(D) = D$, $a \in \{1,2,3\}$. Thus M is semi-invariant submanifold of a manifold $\tilde{M}$ with a Sasakian 3-structure if

$$TM = D \otimes D^{\perp} \otimes \{\xi\},$$

where D, $\{\xi\}$ and $D^{\perp}$ are the above distributions. We note that $D^{\perp}$ is not anti-invariant distribution (see (1.4b)).

From the general theory of Riemannian submanifolds, recall the Gauss and

Weingarten formulae

$$a)\ \tilde{\nabla}_X Y = \nabla_X Y + h(X,Y),$$

$$b)\ \tilde{\nabla}_X N = -A_N X + \nabla_X^{\perp} N,$$
$$\forall X, Y \in \Gamma(TM),\ N \in \Gamma(TM^{\perp}), \qquad (1.5)$$

where h is the second fundamental form of M, A_N is the shape operator with respect to the normal section N, ∇ and $\nabla^{\perp}$ are the induced connections by $\tilde{\nabla}$ on TM and $TM^{\perp}$ and xx, respectively. The Codazzi equation is given by

$$g(K(X,Y)Z, N) = g((\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z)N),$$
$$\forall X, Y, Z \in \Gamma(TM), N \in \Gamma(TM^{\perp}). \qquad (1.6)$$

It is known that if $\{e_i\}\ i = 1, \dots, m$ is an orthonormal basis of $\Gamma(TM)$, then the mean curvature vector field of M, denoted by H, is given by

$$H = \frac{1}{m}\sum_{i=1}^{m} h(e_i, e_i).$$

The submanifold M is called totally contact umbilical if the second fundamental form h of M is expressed as follows

$$h(X,Y) = \sum_{a}^{3}\big(g(f_a X, f_a Y)H + \eta_a(X)h(Y,\xi_a)$$
$$+ \eta_a(Y)h(X,\xi_a)\big),\ \forall X, Y \in \Gamma(TM)\ (1.7)$$

If H = 0 and (1.7) holds, then M is called totally contact geodesic submanifold of $\tilde{M}$.

It is known that any sphere of a Euclidean space is totally umbilical and has positive constant curvature. Also we recall that M is an extrinsic sphere of $\tilde{M}$ if it is totally contact umbilical and has parallel the mean curvature vector field $H \neq 0$, that is,

$$\nabla_X^{\perp} H = 0,\ \forall X \in \Gamma(TM).$$

Finally we recall some properties of semi-invariant submanifolds of a manifold $\tilde{M}$ with a Sasakian 3-structure, for later use (see [2])

Proposition. 1.1. *Let M be a semi-invariant submanifold of a manifold with a Sasakian 3-structure. Then*

a) $h(X, \xi_a) = 0;$

b)

$$h(Z, \xi_a) = -f_a Z,\ \forall X \in \Gamma(D),\ Z \in \Gamma(f_a(\mu^{\perp}))\ (1.8)$$

Also we see that if M is totally contact umbilical then

$$(\nabla_X h)(Y,Z) = 3g(Y,Z)\nabla_X^{\perp} H, \qquad (1.9)$$

if Y and Z belong to $\Gamma(D)$ and $X \in \Gamma(TM)$

Main Results

Let M be a real m-diminsional submanifold of a 2n+1-dimensional manifold $\tilde{M}$ with a Sasakian 3-structure. It was proved (see [2]) that if M is totally contact umbilical semi-invariant proper submanifold ($\dim D > 0$; $\dim D^{\perp} > 0$), with $s = \dim \mu_x^{\perp} > 1$, $x \in M$ then M must be totally contact geodesic. Then it remains to study the cases s = 0 and s = 1. To this end we first prove the following general lemma.

Lemma. 2.1. *Let M be a totally contact umbilical semi-invariant submanifold of a manifold $\tilde{M}$ with a Sasakian 3-structure and $D \neq \{0\}$. Then* *the mean curvature vector field H of M is a global section of* $\Gamma(\mu^{\perp})$.

Proof. Let $X \in \Gamma(D)$ a unit vector field and $N \in \Gamma(\mu)$. By using (1.1g), (1.2a), (1.5a) and (1.7) we deduce that

$$g(H,N) = g(g(X,X)H,N) = g(\tilde{\nabla}_X X, N)$$
$$= g(\tilde{\nabla}_X f_a X - (\tilde{\nabla}_X f_a)X, f_a N) = g(h(X, f_a X), f_a N)$$
$$= g(X, f_a X) g(H, f_a N) = 0$$

which proves our assertion.

Now we see that if s = 0, then H = 0 and M is totally contact geodesic. Next, because M is not totally contact geodesic and it is supposed to be connected, then let $\alpha = \|H\| \neq 0$. Denote

$$\text{a) } U = \frac{1}{\alpha} H, \quad \text{b) } W_a = f_a U, \ a \in \{1,2,3\}. \qquad (2.1)$$

Lemma. 2.2. *Let M be a totally contact umbilical semi-invariant submanifold of a manifold* $\tilde{M}$ *with a Sasakian* 3-structure. Then we have

$$\nabla^{\perp}_X H \in \Gamma(\mu^{\perp}), \ \forall X \in \Gamma(TM).$$

Proof. Let $X \in \Gamma(TM)$ and $N \in \Gamma(\mu)$ Now by using Lemma 2.1 we have $H \in \Gamma(\mu^{\perp})$. By using (1.1g), (1.2), (1.6b) and (1.7) we infer that,

$$g(\nabla^{\perp}_X H, N) = g(\tilde{\nabla}_X f_a H - (\tilde{\nabla}_X f_a)H, f_a N) =$$
$$g(h(X, f_a H), f_a N) = g(X, f_a H) g(H, f_a N) = 0.$$

Therefore our assertion is proved.

Now we prove the main result of the paper

Theorem. 2.1. *Let M be a proper totally contact umbilical semi-invariant submanifold of a manifold with a Sasakian 3-structure, such that* $\dim \mu^{\perp}_x = 1$, *for any* $x \in M$ *and* $H \neq 0$. *Then M is an extrinsic sphere.*

Proof. Let $X \in \Gamma(TM)$, $Y \in \Gamma(D)$. By using (1.3a) and (2.1b) we infer that

$$g(K(W_1, X)f_1 Y, U) = g(f_1 K(W_1, X)Y + g(X,Y)U, U)$$

$$= g(X,Y) - g(K(W_1, X)Y, W_1). \qquad (2.2)$$

On the other hand, using (1.6) and (1.9) we deduce that

$$g(K(W_1, X)f_1 Y, U) = g((\nabla_{W_1} h)(X, f_1 Y) - (\nabla_X h)$$
$$(W_1, f_1 Y)U) = 3g(X, f_1 Y) g(\nabla^{\perp}_{W_1} H, U) - 3g(W_1, f_1 Y)$$
$$g(\nabla^{\perp}_X H, U) = 3g(X, f_1 Y) g(\nabla^{\perp}_{W_1} H, U). \qquad (2.3)$$

The relations (2.2) and (2.3) imply

$$g(X,Y) - g(K(W_1, X)Y, W_1)$$
$$= 3g(X, f_1 Y) g(\nabla^{\perp}_{W_1} H, U) \qquad (2.4)$$

But, using the symmetry properties of the tensors g, K and f_1 with respect to g, we get $g(\nabla^{\perp}_{W_1} H, U) = 0$ which together with Lemma 2.2, imply $\nabla^{\perp}_Z H = 0$, $Z \in \Gamma(D^{\perp})$. Next, let $X \in \Gamma(D)$ be a unit vector field. By using (1.1e), (1.1g) (1.6) and (1.9) we infer that

$$g(K(f_1 X, f_2 X) f_3 X, U) = g((\nabla_{f_1 X} h)(f_2 X, f_3 X)$$
$$-(\nabla_{f_2 X} h)(f_1 X, f_3 X), U) = 3g(f_2 X, f_3 X) g(\nabla^{\perp}_{f_1 X} H, U)$$
$$-3g(f_1 X, f_3 X) g(\nabla^{\perp}_{f_2 X} H, U) = 0 \qquad (2.5)$$

On the other hand, using (1.1a), (1.1c), (1.3a), (1.3b), (1.6) and (1.9) we obtain

$$g(K(f_1 X, f_2 X) f_3 X, U) = -g(f_1 K(X, f_3 X) f_2 X, U)$$
$$= -g(K(X, f_3 X) f_3 X), U) = g((\nabla_{f_3 X} h)(X, f_3 X)$$
$$-g((\nabla_X h)(f_3 X, f_3 X) = -3g(X,X) g(\nabla^{\perp}_X H, U) \qquad (2.6)$$

Now the relations (2.5), (2.6) and Lemma 2.2, imply $\nabla^{\perp}_X H = 0$, $\forall X \in \Gamma(D)$. Taking again $X \in \Gamma(D)$ a unit vector field and using (1.6), (1.7) and (1.8a), we deduce that

$$g(K(\xi_1, X)X, U) = g((\nabla_{\xi_1} h)(X,X)$$

$$(\nabla_X h)(\xi_1, X), U) = g(\nabla^{\perp}_{\xi_1} H, U) \qquad (2.7)$$

Taking into account (1.3c), the fact that $U \in \Gamma(\mu^{\perp})$, from (2.7) and Lemma 2.2 we get $\nabla^{\perp}_{\xi_1} H = 0$. Finally we proved that $\nabla^{\perp}_X H = 0$, $\forall X \in \Gamma(TM)$ The proof is complete.

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