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Differentiable manifold with a [F1,F2] (5,1) - structure

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Abstract

The definition of $[F_1, F_2]$ (5,1) - structure manifold is given. It is approved that there are two distributions L and N on the manifold such that F_j^2, $j = 1,2,3$ act an L as almost complex operators, and act on N as null operators. It is proved that an invariant submanifold of a $[F_1, F_2]$ (5,1)- structure manifold inherites such a structure.

Key words: Polynomial G-structures, integrability, invariant distributions.

Multiplicidad diferenciable con una estructura [F1,F2] (5,1)

Resumen

La definición de multiplicidad de estructura $[F_1,F_2]$ (5,1) es dada.Se prueba que hay dos distribuciones L y N sobre la multiplicidad tales que F_j^2, = 1,2,3 actúan en L como operadores casi complejos, y actúan sobre N como operadores nulos. Se prueba que una submultiplicidad invariante de una multiplicidad de estructura $[F_1,F_2]$ (5,1) hereda una estructura tal.

Palabras clave: Estructura G polinomial, integrabilidad, distribuciones invariantes.

Introduction

A C^∞ manifold on which there exists a C^∞ tensor field $F \neq 0$ of type (1,1) such that

$F^2 = -I$, I is the identity tensor (1)

is called an almost complex manifold with almost complex structure [1].

If we have on M three complex structures F_1, F_2, and F_3 such that

(a) $F_i^2 = -I$, $i = 1,2,3$ (2)

(b) $F_3 = F_1F_2 = -F_2F_1$,
$F_2 = F_3F_1 = -F_1F_3$,
$F_1 = F_2F_3 = -F_3F_2$ (3)

then $\{F_1, F_2, F_3\}$ are said to define an almost quarternion 3-structures on M. If rank $(F_j) = r$, everywhere on M, then dimension $M = 4r$ [2].

If (1) is replaced by

(a) $F^3 + F = 0$,
(b) rank $(F) = r \leq n$ (4)

where n is the dimension of M, then we say that we have an F-structure manifold with structure [3].

If we have on M two F-structures, F_1 and F_2, such that

(a) $F_1^3 + F_1 = 0$, (b) $F_2^3 + F_2 = 0$,
(c) $F_1^2 = F_2^2$, (d) $F_3 = F_1F_2 = -F_2F_1$ (5)

then M is called an $\{F_1, F_2\}$ – structure manifold with $\{F_1, F_2\}$ – structure. It is proved that [4]:

(a) $F_3^3 + F_3 = 0$, (b) $F_1 = F_2F_3 = -F_3F_2$,
(c) $F_2 = F_3F_1 = -F_1F_3$, (d) $F_1^2 = F_2^2 = F_3^2$ (6)

If (4) is replaced by

$$F^5 + F_1 = 0 \tag{7}$$

then M is called an $F(5,1)$ - structure manifold with $F(5,1)$ - structure [5].

On an $F(5,1)$ - structure manifold M, the operators

$$\text{(a)}\quad \ell = -F^4, \qquad \text{(b)}\quad m = F^4 + I \tag{8}$$

Applied to M_p for each $P \in M$ are complementary projection operators. If rank $(F) = r$ everywhere on M, ℓ and m define two differentiable complementary distributions L and N on M of dimensions r and n-r.

Suppose that V is an h-dimensional submanifold of M with immersion

$$b : V \to M \tag{9}$$

Let B be the Jacobian map (B is a linear transformation induced by b) such that a vector field X in V at $P \in V$, $\mapsto BX$ in M at $b(P) \in M$. Let C_x, $x = 1,2,...,n$-h be the field of normals to V [6].

2. [F1,F2] (5,1)-structure manifold

Definition

Let M be a C^∞ n-dimensional manifold on which there are two (5,1)-structures F_i, i =1,2, such that

(a) $F_i^5 + F_i = 0$, (b) $F_1^2 = F_2^2$

(c) Define $F_3 = aF_1F_2 = -aF_2F_1$ (10)

where $a = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\sqrt{-1}$ (i.e. $a^4 = -1$). The we say that we have an $[F_1,F_2]$ (5,1)-structure manifold with $[F_1,F_2]$ (5,1)-structure.

Theorem 1

Let M be an $[F_1,F_2](5,1)$ - structure manifold. Then

(a) $F_3^5 + F_3 = 0$ (b) $F_1^4 = F_2^4 = F_3^4$

(c) $F_2 = aF_3F_1 = -aF_1F_3$, $F_1 = aF_2F_3 = -aF_3F_2$

(d) $F_1^2 = F_2^2 = F_3^2$ (11)

Proof

Using (10) (a) and (c) we have

$$\begin{aligned} F_3^5 &= a^5(F_1F_2)^5 = -a\left[F_1F_2F_1F_2F_1F_2F_1F_2F_1F_2\right] \\ &= -a\left[F_1^2F_2F_1F_2F_1F_2F_1F_2^2\right] = a\left[F_1^3F_2F_1F_2F_1F_2^3\right] \\ &= a\left[F_1^4F_2F_1F_2^4\right] = -aF_1^5F_2^5 = -aF_1F_2 = -F_3 \end{aligned}$$

which is (11) (a)

$$F_1^4 = F_1^2F_2^2 = F_2^2F_2^2 = F_2^4$$

$$\begin{aligned} F_3^4 &= a(F_1F_2)^4 = -\left[F_1F_2F_1F_2F_1F_2F_1F_2\right] \\ &= \left[F_1^2F_2F_1F_2F_1F_2^2\right] = \left[F_1^3F_2F_1F_2^3\right] = -F_1^4F_2^4 \\ &= -F_1^8 = F_1^4 = F_2^4 \end{aligned}$$

which is (11) (b)

Using (10) we have

$$F_1 = -F_1^5 = -F_1^4F_1 = -F_2^4F_1 = \frac{1}{a}F_2^3F_3$$

$$\Rightarrow aF_1 = F_2^3F_3 \tag{12}$$

$$\begin{aligned} aF_1F_3 &= aF_1^5F_3^5 = aF_1F_1^4F_3^4F_3 \\ &= aF_1F_2^8F_3 = -aF_1F_2^4F_3 = -F_3F_2^3F_3 = -F_3(aF_1) \\ &= -aF_3F_1 \end{aligned}$$

$$\therefore F_1F_3 = -F_3F_1 \tag{13}$$

(13), (10) and (11) (a) give

$$F_3F_1 = -F_1F_3 = -F_1^5F_3^5 = -F_1F_2^3F_3 = F_1F_2^4F_3 = F_1F_3F_2^4$$

$$= -F_3F_1F_2^4,$$

where $F_2^4F_3 = F_3F_2^4$

$$\therefore F_3F_1 = -F_3F_1F_2^4 \tag{14}$$

$$\text{But } F_2 = -F_2F_2^4 \tag{15}$$

(14) and (15) suggest that $F_2 = kF_3F_1$, k being a constant

$$F_2^4 = k^4(F_1F_3)^4 = k^4F_1^4F_3^4 = k^4F_2^8 = -k^4F_2^4$$

$$\therefore k^4 = -1 = a^4.$$

$$\text{Put } F_2 = aF_3F_1 = -aF_1F_3 \tag{16}$$

Similarly we can prove that

$F_2F_3 = F_3F_2$, and we have

$$F_1 = aF_2F_3 = -aF_3F_2$$

Which is (11) (c)

Premultiplying $F_3 = aF_1F_2$ by F_3 we get, from (16)

$$F_3^2 = aF_3F_1F_2 = F_2^2$$

which give us (11) (d).

Theorem 2

On an $[F_1,F_2]$ (5,1) – structure manifold M we have

$$\text{rank}(F_1) = \text{rank}(F_2) = \text{rank}(F_3) \quad (17)$$

Assume that rank $(F_i) = r =$ constant, all over M.
Put

$$\text{(a)}\ \ell = -F_1^4 \qquad \text{(b)}\ m = F_1^4 + I \quad (18)$$

then

$$\text{(a)}\ \ell = -F_2^4 = -F_3^4 \quad \text{(b)}\ \ell = F_2^4 + I = F_3^4 + I \quad (19)$$

This means that we have two differentiable distributions L and N on M.

Proof

From (10) (c) and (11) (c) we have

$$\text{rank}(F_1) \le \min[\text{rank}(E_2), \text{rank}(F_3)]$$

$$\text{rank}(F_2) \le \min[\text{rank}(F_3), \text{rank}(F_1)]$$

$$\text{rank}(F_3) \le \min[\text{rank}(F_1), \text{rank}(F_2)]$$

This proves (17)
(19) is obvious.

Theorem 3

On an $[F_1, F_2]$ (5,1) – structure manifold, let $J_j = F_j^2$ then we have

$$\text{(a)}\ F_j\ell = \ell F_j = F_j \quad \text{(b)}\ J_j\ell = \ell J_j = J_j \quad \text{(c)}\ J_j^2\ell = -\ell$$
$$\text{(d)}\ F_jm = mF_j = 0 \quad \text{(e)}\ J_j^2m = 0, \quad j = 1,2,3 \quad (20)$$

Proof

We have $F_1^4 = F_j^4$. Using $F_j^5 + F_j = 0, j=1,2,3$
We have

$$F_j\ell = F_jF_1^4 = -F_jF_j^4 = -F_j^5 = F_j$$

similarly, we have $\ell F_j = F_j$
This proves (20) (a)
Using (10) (a), and (11) (a) and (b) we have

$$J_j\ell = F_j^2F_1^4 = -F_j^6 = F_j^2 = J_j$$

similarly we have $\ell J_j = J_j$
This proves (20) (b).
Using (10) (a), (11) (a) and (b), we have

$$J_j^2\ell = -F_j^4F_1^4 = -F_j^8 = F_j^4 = -\ell$$

This proves (20) (c).
Using (10) (a) and (11) (a) and (b), we have

$$F_jm = F_j(F_1^4 + I) = F_j^5 + F_j = 0$$

similarly we have $mF_j = 0$. This proves (20) (d).
Using (10) (a) and (11) (a) and (b), we have

$$J_j^2m = -F_j^4(F_1^4 + I) = F_j^8 + F_j^4 = 0$$

similarly we have $mJ_j^2 = 0$

Theorem 4

On an $[F_1,F_2]$ (5,1) – structure manifold M, F_j^2 ($j = 1,2,3$) act on L as almost complex structures and on N as null operators.

Proof

Suppose that $X \in L$ i.e. $\ell X = X$. Then

$J_j\ell X = J_jX = \ell J_jX$, where we used (20) (b).

This means that $J_j X \in L$. We cannot have $J_jX = 0$, because if we have $J_jX = 0$, then $J_j^2X = 0$ or $-\ell X = 0$, which contradicts our assumption.

(20) (c) states that $J_j^2\ell = -\ell$, this means that J_j act on L as almost complex operators. Suppose that $Y \in N$, i.e. $mY = Y$. From (20) (b) we have

$J_jY = J_j\ell Y = 0$, i.e. J_j act on N as null operators.

Invariant Submanifolds

Theorem 5

Let V be a C^∞ m – dimensional submanifold of a C^∞ n-dimensional manifold M, let F_i, $i = 1,2$ be two C^∞ tensor fields of type (1,1) on M. Write $F_i(BX)$, as the sum of tangential and normal parts, where X is a vector field in V.

$$F_j(BX) = B(f_jX) + P_j^x(X)C_x,\ j = 1,2 \tag{21}$$

where f_i are two C^∞ tensor fields of type (1.1) on V.

Write $F_i(C_x)$ as the sum of tangential and normal parts

$$F_j(C_x) = -BP_{jx} + a_{jx}^yC_y \tag{22}$$

Then

$$\begin{aligned}F_j^2(BX) &= B(F_j^2X) + P_j^x(F_jX)C_x + P_j^x(X)F_j(C_x)\\ &= B(F_j^2X) + BG_{j1}(X) + H_{j1}^y(X)C_y\end{aligned} \tag{23}$$

$$\begin{aligned}F_j^3(BX) &= B(f_j^3X) + P_j^x(f_j^2X)C_x + P_j^x(f_jX)F_j(C_x)\\ &\quad +P_j^x(X)F_j^2(C_x)\\ &= B(F_j^3X) + BG_{j2}(X) + H_{12}^y(X)C_y\end{aligned} \tag{24}$$

$$\begin{aligned}F_j^4(BX) &= B(f_j^4X) + P_j^x(f_j^3X)C_x + P_j^x(f_j^2X)F_j(C_x)\\ &\quad +P_j^x(f_jX)F_j^2(C_x) + P_j^x(x)F^3(C_x)\\ &= B(F_j^4X) + BG_{j3}(X) + H_{j3}^y(X)C_y\end{aligned} \tag{25}$$

$$\begin{aligned}F_j^5(BX) &= B(f_j^5X) + P_j^x(f_j^4X)C_x + P_j^x(f_j^3X)F_j(C_x)\\ &\quad +P_j^x(f_j^2X)F_j^2(C_x) + P_j^x(f_jX)F_j^3(C_x)\\ &\quad +P_j^x(X)F_j^4(C_x)^n\\ &= B(f_j^5X) + BG_{j4}(X) + H_{j4}^y(X)C_y\end{aligned} \tag{26}$$

where $B(F_j^{i+1}X) + BG_{ji}(X)$ and $H_{ji}^y(X)C_y$ are the tangential and normal parts to V respectively. $j = 1,2$, $i = 1,2,3,4$

$$\begin{aligned}F_j^2(C_x) &= -B(f_jP_{jx}) - P_j^y(P_{jx})C_y + a_{jx}^yF(C_y)\\ &= -B(f_jP_{jx}) + BT_{j1}(X) + R_{j2}^y(X)C_y\end{aligned} \tag{27}$$

$$\begin{aligned}F_j^3(C_x) &= -B(f_j^2P_{jx}) - P_j^y(f_jP_{jx})C_y - P_j^y(P_{jx})F_j(C_y)\\ &\quad +a_{jx}^yF_j^2(C_y) = -B(f_j^2P_{jx}) + BT_{j2}(X)\\ &\quad +R_{j2}^y(X)C_y\end{aligned} \tag{28}$$

$$\begin{aligned}F_j^4(C_x) &= -B(f_j^3P_{jx}) - P_j^y(f_j^2P_{jx})C_y - P_j^y(f_jP_{jx})F_j(C_y)\\ &\quad -P_j^y(P_{jx})F_j^2(C_y) + a_{jx}^yF_j^3(C_y) = -B(f_j^3P_{jx})\\ &\quad +BT_{j3}(X) + R_{j3}^y(X) = C_y\end{aligned} \tag{29}$$

$$\begin{aligned}F_j^5(C_x) &= -B(f_j^4P_{jx}) - P_j^y(f_j^3P_{jx})C_y - P_j^y(f_j^2P_{jx})F_j(C_y)\\ &\quad -P_j^y(f_jP_{jx})F_j^2(C_y) - P_j^y(P_{jx})F_j^3(C_y)\\ &\quad +a_{jx}^yF_j^4(C_y)\\ &= -B(f_j^4P_{jx}) + BT_{j4}(X) + R_{j4}^y(X)C_y\end{aligned} \tag{30}$$

where $-B(f_j^iP_{jx}) + B.T_{ji}(X)$ and $R_{ji}^y(X)C_y$ are the tangential and normal parts to V respectively, $i = 1,2,3,4$ and $j = 1,2$.

$$\begin{aligned}F_1F_2(BX) &= B(f_1f_2X) + P_1^x(f_2X)C_x + P_2^x(X)F_1(C_x)\\ &= B(f_1f_2X) + BS_{12}(X) + U_{12}^y(X)C_y\end{aligned} \tag{31}$$

$$\begin{aligned}F_1F_2(BX) &= B(f_2f_1X) + P_2^x(f_1X)C_x + P_1^x(X)F_2(C_x)\\ &= B(f_2f_1X) + BS_{21}(X) + U_{21}^y(X)C_y\end{aligned} \tag{32}$$

where $BS_{12}(X)$, $BS_{21}(X)$, are the tangential parts and $U_{12}^y(X)$, $U_{21}^y(X)$ are the normal parts.

Proof

Premultiplying (21) by F_i and using (21) again we get (23).

Premultiplying (23) by F_i and using (21) we get (24). Similarly we can get (25) and (26).

Premultiplying (22) by F_i and using (21) we get (27). Similarly we can get (28), (29) and (30).

From (21) we have

$$F_2(BX) = B(f_2X) + P_2^x(X)C_x$$

Premultiplying by F_1 and (21) we get (31). Similarly we have (32).

Theorem 6

Let M be an $[F_1\ F_2]$ (5,1) – structure manifold, then

$$F_j^5X + f_j(X) + G_{j4}(X) = 0 \tag{33}$$

$$H_{j4}^y(X) + P_j^y(X) = 0 \tag{34}$$

$$f_1^2X - f_2^2X + G_{11}(X) - G_{21}(X) = 0 \tag{35}$$

$$H_{11}^y(X) - H_{21}^y(X) = 0 \tag{36}$$

$$f_1f_2X + f_2f_1X + S_{12}(X) + S_{21}(X) = 0 \tag{37}$$

$$U_{12}^y(X) + U_{21}^y(X) = 0 \tag{38}$$

Proof

M is an $[F_1,F_2]$ (5,1) – structure manifold. Then

$$F_i^5(BX) + F_i(BX) = 0$$

From (26) and (21) we have

$$B(f_i^5X)+BG_{i4}(X)+H_{i4}^y(X)C_y+B(f_iX)+P_i^yC_y=0$$

This gives (33) and (34)

From $F_1^2(BX)=F_2^2(BX)$ we have

$$Bf_1^2X+BG_{11}(X)+H_{11}^y(X)C_y=Bf_2^2X+BG_{21}(X)+H_{21}^y(X)C_y$$

This gives (35) and (36)

From

$$F_3(BX)=aF_1F_2(BX)=-aF_2F_1(BX)$$

we have

$$B(f_1f_2X)+BS_{12}(X)+U_{12}^y(X)C_y=-B(f_2f_1X)-BS_{21}(X)-U_{21}^y(X)C_y$$

This gives (37) and (38)

Theorem 7

Let V be an invariant submanifold of M with respect to both F_i, $i=1,2$. Then

$$F_j^i(BX)=B(f_j^iX),\ j=1,2 \text{ and } i=2,3,4,5 \quad (39)$$

$$F_j^i(C_x)=-B(f_j^{i-1}P_{jx})+a_{jx}^yF^{i-1}(C_y) \quad (40)$$

$$F_1F_2(BX)=B(f_1f_2X),\quad \text{(b)}\ F_2F_1(BX)=B(f_2f_1X) \quad (41)$$

Proof

That V is an invariant submanifold means that

$$F_j(BX)=B(f_jX)\Rightarrow P_j^x=0,\ j=1,2$$

This in (23), (24), (25) and (26) give (39) also put $p_j^x=0$ in (27), (28), (29) and (30) give (40), with $j=1,2$ and $i=2,3,4,5$. Again if we put $p_j^x=0$ in (31) and (32) we get (41).

Theorem 8

Let V be an invariant submanifold with respect to both F_i, $i=1,2$ of an $[F_1, F_2]$ (5,1) – structure manifold M, then

$$f_i^5+f_i=0 \quad (42)$$

$$f_1^2=f_2^2 \quad (43)$$

$$f_1f_2=-f_2f_1 \quad (44)$$

and V itself is an $[f_1,f_2]$ (5,1) – structure manifold.

Proof

Putting $j=2$ and 5 in (39) and using the fact that

$$F_i^5(BX)+F_i(BX)=0,\ i=1,2$$

and $F_1^2(BX)=F_2^2(BX)$

we get (42) and (43)

$$F_1F_2(BX)=-F_2F_1(BX)$$

(41) gives (44)

Let $f_3=af_1f_2=-af_2f_1$, $a^4=-1$

We have V itself is an $[f_1,f_2]$ (5,1) – structure manifold.

References

1. Yano K.: Differential Geometry On Complex And Almost Complex Spaces, Pergamon Press N.Y. 1965.
2. Ishihara S.: Quaternion Kahler Manifolds, J. Diff. Geom. 9(4), (1974), 433-500.
3. Yano K.: On A Structure Defined By A Tensor Field Of Type (1,1) Satisfying $F^3+F=0$, Tensor N.S. 14, (1963), 99-109.
4. Hashimoto S.: On A Differentiable Manifold M^n Admitting Tensor Fields (F,G) Of Type (1,1) Satisfying $F^3+F=0$, $G^3+G=0$, $FG=-GF$, And $F^2=G^2$. Tensor N.S. 15 (1964), 267-274.
5. Geuli-Andreou F.: On A Structure Defined by A Tensor Field F Of Type (1,1) Satisfying $F^5+F=0$. Tensor N.S. 36 (1932), 99-109.
6. Hicks N.J.: Notes On Differential Geometry Van Nostrand, Princeton 1965.

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