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Certain results associated with the generalized Riemann zeta functions

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Abstract

In this paper we derive new (linear and bilateral) generating functions involving certain families of hypergeometric functions and the Riemann zeta function (or the generalized zeta function). A connection with the Lambert transform is given, and a generalized Lambert transform is introduced. An inversion formula of this transform is obtained and its relationship with the generalized zeta function is also depicted.

Key words: Riemann zeta function, generating function, Hurwitz zeta function, Lambert transform, Laguerre polynomials, Gauss hypergeometric function, Mellin transform, generalized hypergeometric function, Mellin inversion theorem.

Ciertos resultados asociados con las funciones zeta de Riemann

Resumen

En este trabajo derivamos funciones generatrices nuevas (lineales y bilaterales) que involucran ciertas familias de funciones hipergeométricas y la función zeta de Riemann (o la función zeta generalizada). Se presenta una conexión con la transformada de Lambert y se introduce una transformada generalizada de Lambert. Se obtiene una fórmula de inversión de esta transformada y también se describe su relación con la función zeta generalizada.

Palabras claves: Función zeta de Riemann, función generatriz, función zeta de Hurwitz, transformada de Lambert, polinomios de Laguerre, función hipergeométrica de Gauss, transformada de Mellin, función hipergeométrica generalizada, teorema de inversión de Mellin.

Introduction

The generalized (Hurwitz's) zeta function is defined by [1, p. 24]:

$$\zeta(s,a) = \sum_{n=0}^{\infty} (a+n)^{-s} \qquad (1.1)$$
$$(\Re(s) > 1;\ a \neq 0,-1,-2,\cdots),$$

so that, evidently,

$$\zeta(s,1) = \sum_{n=1}^{\infty} n^{-s} = \zeta(s), \qquad (1.2)$$

where $\zeta(s)$ is the Riemann zeta function. The function $\Phi(z,s,a)$ extends (1.1) further, and is defined by [1, p. 27, Equation (1)]

$$\Phi(z,s,a) = \sum_{n=0}^{\infty} (a+n)^{-s} z^n \quad (\Re(a) > 0;\ |z| < 1). \qquad (1.3)$$

Equivalently, the function $\Phi(z,s,a)$ has the integral representation:

$$\Phi(z,s,a) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-at}(1 - ze^{-t})^{-1}\, dt, \tag{1.4}$$

provided that $\Re(a) > 0$ (and either $|z| \leq 1$, $z \neq 1$, and $\Re(s) > 0$, or $z = 1$ and $\Re(s) > 1$).

Our object in the present paper is to obtain certain types of linear as well as bilateral generating functions involving the function $\Phi(z,s,a)$ defined by (1.3). A generalized Lambert transform is introduced and its inversion is obtained, and further expressions in terms of the function $\Phi(z,s,a)$ are derived from the Lambert transform of a general system of polynomials.

Generating Functions

From the definition (1.3), and the binomial expansion

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} z^n = (1 - z)^{-\lambda} \quad (|z| < 1), \tag{2.1}$$

we readily have the generating function:

$$\sum_{n=0}^{\infty} (\lambda)_n \Phi(x,\lambda + n,a) \frac{t^n}{n!} = \Phi(x,\lambda,a - t) \tag{2.2}$$

$$(|t| < |a|;\ \lambda \neq 1),$$

where, and in what follows, $(\lambda)_n = \Gamma(\lambda + n)/\Gamma(\lambda)$.

The generating function (2.2) can be extended further. Indeed, in terms of Gauss's hypergeometric function [1, p. 56], it follows from (1.3) that

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n(\mu)_n}{(\nu)_n} \Phi(x,\lambda + \mu - \nu + n,a) \frac{t^n}{n!} =$$

$$\sum_{n=0}^{\infty} x^n(n + a)^{\nu-\lambda-\mu}\, {}_2F_1\left[\begin{matrix}\lambda,\mu; \\ \nu;\end{matrix}\ \frac{t}{n + a}\right], \tag{2.3}$$

provided that $|t/a| < 1$ and $\Re(\lambda + \mu) > \Re(\nu) > 0$. More generally, (2.3) may be extended in the form:

$$\sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_n}{\prod_{j=1}^{q} (b_j)_n} \Phi(x,\omega + n,a) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} x^n(n + a)^{-\omega}\, {}_pF_q\left[\begin{matrix}(a_p); \\ (b_q);\end{matrix}\ \frac{t}{n + a}\right], \tag{2.4}$$

provided that $p \leq q$, $|t| < |a|$, and $\Re(\omega) > 0$. Here (a_p) abbreviates the array of p parameters

$$a_1, \cdots, a_p,$$

with similar interpretations for (b_q) *etcetera*, ${}_pF_q$ being a generalized hypergeometric function with p numerator and q denominator parameters [1, Chapter 4].

Formulas (2.3) and (2.4) may be viewed (for example) as generating functions for the families of hypergeometric functions occurring on their right-hand sides.

Next we consider a polynomial set $\{S_n^{(m)}(z)\}_{n=0}^{\infty}$ defined by (*cf.* [3, p. 1, Eq. (1)])

$$S_n^{(m)}(z) = \sum_{j=0}^{[n/m]} \frac{(-n)_{mj}}{j!} C_j z^j \tag{2.5}$$

$$(m \in \mathbb{N};\ n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}),$$

where, and in what follows, $\{C_n\}_{n=0}^{\infty}$ is a suitably bounded sequence of complex numbers. Then, by simple series rearrangement technique and the application of (2.2), we arrive at the following bilateral generating function:

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \Phi(x,\lambda + n,a)\, S_n^{(m)}(y)\, t^n$$

$$= \sum_{n=0}^{\infty} \frac{(\lambda)_{mn}}{n!} C_n\, \Phi(x,\lambda + mn,a - t)\, \{y(-t)^m\}^n, \tag{2.6}$$

provided that $|x| \leq 1$, $|t| < |a|$, and $\lambda \neq 1$.

Example. By specializing the sequence $\{C_n\}$ as follows:

$$C_n = \frac{\prod_{j=1}^{p} (a_j)_n}{m^{mn} \prod_{j=1}^{q} (b_j)_n} \quad (m \in \mathbb{N};\ n \in \mathbb{N}_0), \quad (2.7)$$

we find from (2.6) that

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \Phi(x, \lambda + n, a)\ {}_{m+p}F_q \left[\begin{matrix} \Delta(m; -n), (a_p); \\ (b_q); \end{matrix}\ y \right] t^n$$

$$= \sum_{n=0}^{\infty} \frac{(\lambda)_{mn}}{n!} \frac{\prod_{j=1}^{p} (a_j)_n}{\prod_{j=1}^{q} (b_j)_n} \Phi(x, \lambda + mn, a - t) \left\{ y \left(-\frac{t}{m} \right)^m \right\}^n, \quad (2.8)$$

where $\Delta(m; \lambda)$ denotes the array of m parameters

$$\frac{\lambda}{m}, \frac{\lambda+1}{m}, \cdots, \frac{\lambda+m-1}{m} \quad (m \in \mathbb{N}).$$

In particular, in terms of the classical Laguerre polynomials $L_n^{(\alpha)}(z)$, for $p = 0$, $q = m = 1$, and $b_1 = 1 + \alpha$, (2.8) yields

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{(1+\alpha)_n} \Phi(x, \lambda + n, a) L_n^{(\alpha)}(y)\, t^n$$

$$= \sum_{n=0}^{\infty} \frac{(\lambda)_n}{(1+\alpha)_n} \Phi(x, \lambda + n, a - t) \frac{(-yt)^n}{n!}. \quad (2.9)$$

Our formula (2.2) includes some recent results due to Srivastava ([4, p. 48, Equation (2.2)]; see also [5]) and Miller [2, p. 99, Equation (3.20)].

The Lambert Transform

Let $f(t)$ $(t \geq 0)$ be a continuous function. Also let

$$f(t) = O(e^{\kappa t}) \quad (t \to \infty).$$

Then the Lambert transform of $f(t)$ is defined by

$$\bar{f}(s) = LM\{f(t)\} = \int_0^{\infty} \frac{st}{e^{st} - 1} f(t)\, dt, \quad (3.1)$$

and it converges for all s such that $\Re(s - \kappa) > 0$ It follows easily from (3.1) and (1.1) that

$$LM\{t^{\alpha-1} e^{-\nu st}\} = \frac{\Gamma(\alpha+1)}{s^{\alpha}} \zeta(\alpha + 1, \nu + 1) \quad (3.2)$$

$(\Re(\alpha) > -1;\ \nu \neq -1, -2, -3, \cdots;\ \Re((\nu+1)s) > 0)$.

Also, in view of (3.1) and (2.5), we have

$$LM\{t^{\alpha-1} e^{-\nu st} S_m^{(q)}(xt)\}$$

$$= \sum_{j=0}^{[m/q]} \frac{(-m)_{qj}}{j!} \frac{\Gamma(\alpha + j + 1)}{s^{\alpha+j}} \zeta(\alpha + j + 1, \nu + 1)\, C_j x^j \quad (3.3)$$

A special case of (3.3) of interest is the one which occurs when

$$C_j = \frac{1}{(\alpha+1)_j} \quad (j \in \mathbb{N}_0)$$

and $q = 1$. Thus, in terms of the Laguerre polynomials, (3.3) gives

$$LM\{t^{\alpha-1} e^{-\nu st} L_m^{(\alpha)}(xt)\}$$

$$= \frac{\Gamma(m + \alpha + 1)}{m!\, s^{\alpha}} \sum_{j=0}^{m} \frac{(-m)_j}{j!} \zeta(\alpha + j + 1, \nu + 1) \left(\frac{x}{s} \right)^j, \quad (3.4)$$

where $\Re(\alpha) > -m-1$ $(m \in \mathbb{N}_0)$, $\Re((\nu+1)s) > 0$, and $\nu \neq -1, -2, -3, \cdots$.

The Generalized Lambert Transform

We consider the following generalization of the Lambert transform (3.1):

$$F(s) = GLM\{f(t)\} = \int_0^{\infty} \frac{st}{e^{st} - x} f(t)\, dt, \quad (4.1)$$

provided that $\Re(s) > 0$, $|x| \leq 1$, $f(t) \in \Omega$, and $\Re(\gamma) > -2$, where Ω denotes the class of functions $f(t)$ which are continuous for $t > 0$ and satisfy the order estimates:

$$f(t) = \begin{cases} O(t^{\gamma}) & (t \to 0+) \\ O(t^{\delta}) & (t \to \infty). \end{cases} \quad (4.2)$$

The parameter δ occurring in the order estimates (4.2) is unrestricted, in general, since $\Re(s) > 0$. Obviously, for $x = 1$, (4.1) reduces to the Lambert transform (3.1). If $f(t) = t^{\alpha-1}e^{-vst}$ then we find from (1.4) and (4.1) that

$$GLM\{t^{\alpha-1}e^{-vst}\} = \frac{\Gamma(\alpha+1)}{s^{\alpha}}\,\Phi(x, \alpha+1, v+1). \quad (4.3)$$

Inversión of the Generalized Lambert Transform (4.1).

On applying the Mellin transform [6, p. 46], (4.1) yields

$$\begin{aligned} \phi(k) &= \int_0^{\infty} s^{-k-1} F(s)\, ds \\ &= \int_0^{\infty} s^{-k-1}\left(\int_0^{\infty} \frac{st}{e^{st}-x} f(t)\, dt\right) ds \\ &= \int_0^{\infty} f(t)\left(\int_0^{\infty} \frac{st}{e^{st}-x}\cdot s^{-k-1}\, ds\right) dt \quad (4.4) \\ &= \int_0^{\infty} f(t)\cdot\frac{\Gamma(1-k)}{t^{k}}\,\Phi(x, 1-k, 1)\, dt, \end{aligned}$$

where, in addition to the existence and convergence conditions stated with the definition (4.1), we require that $\Re(k) < 1$ for the convergence of the inner s-integral in (4.4). Now, by the Mellin inversion theorem, we get the following inversion formula for the generalized Lambert transform (4.1):

$$\frac{1}{2}\{f(t+0) + f(t-0)\}$$
$$= \frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty}\{\Gamma(1-\zeta)\,\Phi(x, 1-\zeta, 1)\}^{-1} t^{-\zeta}\phi(\zeta)\, d\zeta \quad \left(\tau > \frac{1}{2}\right), \quad (4.5)$$

provided that $t^{\zeta-1} f(t) \in L(0,\infty)$ and $f(t)$ is of bounded variation in the neighbourhood of the point t, $\phi(\zeta)$ being given by (4.4). The constraint $\tau > \frac{1}{2}$ has been imposed with a view to avoiding zeros of the zeta function involved in (4.5).

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