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On some new properties of the Kontorovich-Lebedev like integral transforms

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Abstract

In this paper, some new properties of the Kontorovich-Lebedev and Lebedev-Shalskaya integral transforms in L_p- spaces are established. Some theorems concerning the mappings and inversions in $L_p(R_+)$ are proved.

Key words: Kontorovich-Lebedev transform, Lebedev-Shalskaya transform, index integrals, Macdonald function.

Algunas nuevas propiedades de las transformadas integrales Kontorovich-Lebedev

Resumen

En el presente trabajo se establecen algunas nuevas propiedades de las transformadas integrales Kontorovich-Lebedev y Lebedev-Shalskaya en los espacios L_p- . Se comprueban algunos teoremas con respecto a las representaciones e inversiones en $L_p(R_+)$.

Palabras claves: Transformada Kontorovich-Lebedev, transformada Lebedev-Shalskaya, integrales índice, función Macdonald.

Introduction

As it is known, the classical one-dimensional integral transforms on half-axis R_+ are of the form

$$g(x) = \int_{R_+} H(x,y)f(y)dy, \tag{1}$$

where $H(x,y)$ is some given function (the kernel of the transform), $f(y)$ is an original in a certain space of function and $g(x)$ is the image of the function $f(y)$. All classical integral transforms may be divided into two classes: the Mellin convolution type transforms (or the Fourier type transforms)

$$g(x) = \int_{R_+} k(xy)f(y)dy, \tag{2}$$

with the kernel $H(x,y) = k(xy)$, which is the function of one variable $z = xy$ and transforms when the kernel, generally speaking, is essentially the function of two variables. We will call the last class of integral transforms the index one, since in some known examples of such transforms the kernel $H(x,y)$ is a special function and variable x is its index (a parameter). We note the most important Kontorovich-Lebedev transform [2]

$$g(x) = \int_0^\infty K_{ix}(y)f(y)dy, \tag{3}$$

with the Macdonald function $K_{ix}(y)$ [4, Vol.2] the Mehler-Fock transform [6]

$$g(x) = \int_0^\infty P_{-1/2+ix}(\cosh y)f(y)dy, \tag{4}$$

with the spherical Legendre function of the first kind $P_{-1/2+i\tau}(\cosh y)$ [4, Vol. 3] and the most general transform pair with the Meijer's G- function as the kernel [6], which contains the formulae (1.3), (1.4), namely

$$g(x) = \int_0^\infty G^{m,n+2}_{p+2,q}\left[y \middle| \begin{matrix} 1-\upsilon+ix,\ 1-\upsilon-ix,(a_p) \\ (b_q) \end{matrix} \right] f(y)dy, \quad (5)$$

$$f(x) = \frac{1}{\pi^2}\int_0^\infty \tau \sinh(2\pi\tau)$$

$$G^{q-m,p-n+2}_{p+2,q}\left[x \middle| \begin{matrix} \upsilon+i\tau, \upsilon-i\tau, -(a_p^{n+1}), -(a_n) \\ -(b_q^{m+1}), -(b_m) \end{matrix} \right] g(\tau)d\tau, \quad (6)$$

where $m, n, p, q \in N, 0 \le n \le p, 0 \le m \le q, \upsilon$ is a complex parameter and $(a_p) = (a_1,\cdots,a_p)$, $(b_q) = (b_1,\cdots,b_q)$, $-(a_p^{n+1}) = (-a_{n+1},\cdots,-a_p)$, $-(b_q^{m+1}) = (-b_{m+1},\cdots,-b_q)$ are the parameters of G-functions.

The Kontorovich-Lebedev transform (1.3) has been investigated in various functional spaces (see [1], [5]). In this paper we shall continue both of its consideration and study of the related index transforms in Lp- spaces. Moreover it will be demonstrated a new technique of their investigations through the properties of the Poisson kernel.

The Kontorovich-Lebedev transform

Let us consider the Lp- properties of the following Kontorovich-Lebedev transform

$$[KL_\alpha f](\tau) = \sinh(\alpha\tau)\int_0^\infty K_{i\tau}(y)f(y)dy \quad (7)$$

where $0<\alpha<\pi/2$, $f(y) \in L_p(R_+)$, $1 \le p \le \infty$. We shall develop the L_p-properties of the transform (7), describing the respective space of functions, which connects with the transform (7). As it is evident from usual Hölder inequality and from the asymptotic behaviour of the Macdonald function

$$K_{i\tau}(y) = O(\log y),\ K_\nu(y) = O(y^{-|Re(\nu)|}),\ Re(\nu) \ne 0,\ y \to 0+, \quad (8)$$

$$K_{i\tau}(y) = O(e^{-y}\sqrt{\pi/(2y)}),\ y \to +\infty, \quad (9)$$

the integral (7) converges absolutely for any $p \ge 1$. Let us consider the space of functions $g(\tau)$, which can be represented by the Kontorovich-Lebedev transform, where the respective function $f(y)$ belongs to $L_p(R_+)$

$$KL_\alpha(L_p) = \{g : g(\tau) = [KL_\alpha f](\tau), f \in L_p(R_+)\},$$
$$0<\alpha<\pi/2,\ p \ge 1 \quad (10)$$

We make use of the following estimate for the Macdonald function $K_{i\tau}(x)$ from [1] which holds for all $\tau > 0$ and $x > 0$

$$|K_{i\tau}(x)| \le C\frac{\tau+1}{\tau}e^{-\delta\tau - x\cos(\delta)}, \quad (11)$$

where C is a positive constant and $0 \le \delta < \pi/2$. Applying the general Minkowski inequality to the integral (7), we find that the operator $[KL_\alpha f]$ is bounded mapping from any space $L_p(R_+)$, $1 \le p \le \infty$ into the space $L_q(R_+)$, $1 \le q \le \infty$. Indeed, we have,

$$\|[KL_\alpha f]\|_{L_q(R_+)} \le \int_0^\infty |f(y)| \left(\int_0^\infty \sinh^q(\alpha\tau)\,|K_{i\tau}(y)|^q d\tau\right)^{1/q} dy$$
$$\le C_1\int_0^\infty |f(y)|\, e^{-y\cos(\delta)} \left(\int_0^1 (\tau+1)^q e^{-\delta q\tau}d\tau + \int_1^\infty \left(\frac{\tau+1}{\tau}\right)^q e^{q\tau(\alpha-\delta)}d\tau\right)^{1/q} dy$$
$$\le C_\delta \|f\|_{L_p},\ 1 \le p \le \infty, \quad (12)$$

where C_1, C_δ are positive constants, $\delta > \alpha$. In the last inequality we applied additionally the Hölder inequality.

In order to describe the introduced space $KL_\alpha(L_p)$ let us consider the following operator

$$(I_\varepsilon^\alpha g)(x) = \frac{2}{\pi^2 x^{1-\varepsilon}}\int_0^\infty \frac{\tau\sinh((\pi-\varepsilon)\tau)}{\sinh(\alpha\tau)} K_{i\tau}(x)g(\tau)d\tau, \quad (13)$$

where $\varepsilon \in (0,\pi)$.

Theorem 1

For functions $g(\tau) = [K L_\alpha f](\tau)$ which are represented by the Kontorovich-Lebedev transform (7) with the density $f(y) \in L_p(R_+)$, $1 \le p \le \infty$, the operator (13) has the following form

$$(I_\varepsilon^\alpha g)(x) = \frac{\sin(\varepsilon)}{\pi x^{-\varepsilon}} \times \int_0^\infty \frac{K_1((x^2+y^2-2xy\cos(\varepsilon))^{1/2})}{(x^2+y^2-2xy\cos(\varepsilon))^{1/2}} y f(y)dy, \; x > 0, \quad (14)$$

where $K_1(z)$ is the Macdonald function with $\nu = 1$.

Proof. Substituting the value of $g(\tau)$ as the Kontorovich-Lebedev transform (7) in the formula (13) and changing order of integration in the absolute convergent iterated integral, we use the following equality (see integral 2.16.51.8 from [4], Vol. 2)

$$\int_0^\infty t \sinh((\pi - \varepsilon)) K_{it}(x) K_{it}(y)dt = \frac{\pi xy \sin(\varepsilon)}{2} \frac{K_1((x^2 + y^2 - 2xy\cos(\varepsilon))^{1/2})}{(x^2 + y^2 - 2xy\cos(\varepsilon))^{1/2}} \quad (15)$$

and the Fubini theorem to obtain the representation (14).

Theorem 2

Let $g(\tau) = [K L_\alpha f](\tau)$, $f(y) \in L_p(R_+)$, $1 \le p \le \infty$. Then

$$f(x) = (I^\alpha g)(x), \quad (16)$$

where $(I^\alpha g)(x)$ is understood as

$$(I^\alpha g)(x) = l.i.m._{\varepsilon \to 0+} (I_\varepsilon^\alpha g)(x), \; x > 0, \quad (17)$$

where the limit in (17) is understood in terms of the norm in L_p. Moreover, the limit in (17) exists almost everywhere on R_+.

Proof. The proof of this theorem follows without difficulties from the definition of the integral (14). Indeed, after changing variable $y = (x(\cos(\varepsilon) + t\sin(\varepsilon))$, we get the following equality

$$(I_\varepsilon^\alpha g)(x) = \frac{1}{\pi} \int_{-\infty}^\infty \frac{1}{t^2+1} f(x(\cos(\varepsilon) + t\sin(\varepsilon)) (\cos(\varepsilon) + t\sin(\varepsilon))R(x,t,\varepsilon)dt, \; \varepsilon \in (0,\pi), \quad (18)$$

where

$$R(x,t,\varepsilon) = \begin{cases} x^{\varepsilon+1}\sin(\varepsilon)(t^2+1)^{1/2} K_1(x\sin(\varepsilon)(t^2+1)^{1/2}), & t \ge -ctg(\varepsilon), \\ 0, & t < -ctg(\varepsilon). \end{cases} \quad (19)$$

It is not difficult to see from the asymptotic behaviour of the Macdonald function $K_1(z)$ that for any $t \in R$, $x \in R_+$, and $\varepsilon \in (0,\pi)$ $|R(x,t,\varepsilon)| < C$, where C is a positive constant that.

$$\lim_{\varepsilon \to 0+} R(x,t,\varepsilon) = 1.$$

Further, we use the approximation properties of the Poisson kernel $P(t) = \frac{1}{\pi}\frac{1}{t^2+1}$ and we estimate the following L_p-norm applying the general Minkowski inequality, namely

$$\| (I_\varepsilon^\alpha g) - f \|_{L_p(R_+)} \le \frac{1}{\pi} \int_{-\infty}^\infty \frac{1}{t^2+1} \| f(x(\cos(\varepsilon) + t\sin(\varepsilon)) (\cos(\varepsilon) + t\sin(\varepsilon))R(x,t,\varepsilon) - f(x) \|_{L_p(R_+)} dt \to 0, \; \varepsilon \to 0+. \quad (20)$$

Indeed, due to the estimate from (18)

$$\| (I_\varepsilon^\alpha g) \|_{L_p(R_+)} < \frac{C}{\pi} \int_{-\infty}^\infty \frac{1}{t^2+1} \| f(x(\cos(\varepsilon) + t\sin(\varepsilon)) \times (\cos(\varepsilon) + t\sin(\varepsilon)) \|_{L_p(R_+)} dt$$
$$< C \| f \|_{L_p(R_+)} \frac{1}{\pi} \int_{-\infty}^\infty \frac{(1+|t|)^{1-1/p}}{t^2+1} dt = C_1 \| f \|_{L_p(R_+)}, \quad 1 \le p < \infty, \quad (21)$$

where C_1 is a positive absolute constant, from the Lebesgue theorem and the continuity of the L_p-norm we get the equality (17). The existence of the limit almost everywhere on R_+ follows from the radial property of the Poisson kernel $P(t) = P(|t|) \in L_1(R_+)$.

From estimate (21), in view of (16), the following inequality is true

$$\| (I_\varepsilon^\alpha g) \|_{L_p(R_+)} \le C_1 \| (I^\alpha g) \|_{L_p(R_+)},$$

$$g \in K\, L_\alpha\, (L_p),\ 1 \le p \le \infty. \tag{22}$$

From theorem 2 follows that $[K L_\alpha f](\tau) \equiv 0, f(y) \in L_p(R_+)$, $1 \le p < \infty$, iff $f(y) \equiv 0$. So, in the space $K L_\alpha (L_p)$ we can introduce a norm by the equality

$$\| g \|_{K L_\alpha (L_p)} = \| f \|_{L_p},\ g = [K L_\alpha f] \tag{23}$$

As it is evident, the space $K L_\alpha (L_p)$ is a Banach one with the norm (23) and is isometric to L_p.

The main theorem of this section describes the space $K L_\alpha (L_p)$ in term of the operators (13).

Theorem 3

An arbitrary function $g(\tau)$ which is defined on R_+ and is extended on R as an odd function belongs to the space $K L_\alpha (L_p)$, $1 \le p \le \infty$, and only if $g(\tau) \in L_r(R_+)$, $1 \le r \le \infty$ and the following condition holds

$$l.i.m._{\varepsilon \to 0+} (I^\alpha_\varepsilon g) \in L_p(R_+). \tag{24}$$

Proof. The necessity of condition (24) follows from the previous theorem 2 and from estimate (12). Let us prove the sufficiency: Let $g(\tau) \in L_r(R_+)$, $g(\tau) = -g(-\tau)$ and condition (24) holds. We show that in this case there is a function $f \in L_p$, such that the equality.

$$g = [K L_\alpha f] \tag{25}$$

holds. From inequality (22), we conclude that $(I^\alpha_\varepsilon g) \in L_p$ for each $\varepsilon \in (0,\pi)$ and we can evaluate the following composition

$$[K L_\alpha (I^\alpha_\varepsilon g)](\tau) = \sinh(\alpha\,\tau) \int_0^\infty K_{i\tau}(y)\, (I^\alpha_\varepsilon g)\,(y) dy. \tag{26}$$

At least for smooth functions with compact support on R_+, whose set is dense in L_r, by substituting (13) in equality (26) and using the value of the integral 2.16.33.2 from [4], Vol. 2 after changing the order of integration, we find

$$[K L_\alpha (I^\alpha_\varepsilon g)](\tau) = g_\varepsilon(\tau) = \sinh(\alpha\,\tau)\ \frac{2^{\varepsilon-2}}{\pi^2 \Gamma(\varepsilon)} \int_0^\infty \frac{t \sinh((\pi-\varepsilon)t)}{\sinh(\alpha\,\tau)} \times \left| \Gamma\left(\frac{\varepsilon + i(t+\tau)}{2}\right) \Gamma\left(\frac{\varepsilon + i(t-\tau)}{2}\right) \right|^2 g(t) dt. \tag{27}$$

In order to prove the validity of equality (27) for all $g \in L_r$ we must prove now the boundedness of the operator in the right side of (27). But as it is not difficult to see from the asymptotic formula for gamma-function the kernel of the integrand in (27) is equal to

$$O(e^{(\pi/2-\alpha)(t-\tau) - \pi/2\,|t-\tau| - \varepsilon t}),\ (t,\tau) \in R_+ \times R_+,\quad \alpha \in (0,\pi/2),\ \varepsilon \in (o,\pi). \tag{28}$$

Hence we have the following estimate

$$| [K L_\alpha (I^\alpha_\varepsilon g)](\tau) | < C_1 e^{(\alpha - \pi/2)\tau} \times \int_0^\infty e^{(\pi/2 - \varepsilon - \alpha)t - \pi/2\,|t-\tau|}\, | g(t) | dt. < C_1 e^{(\alpha - \pi/2 + \delta)\tau} \int_0^\infty e^{(\pi/2 - \varepsilon - \alpha - \delta)t}\, | g(t) | dt, \tag{29}$$

where the value of δ is taken from the interval $(\pi/2 - \alpha - \varepsilon, \pi/2 - \alpha)$. So from estimate (29) with the aid of the Hölder inequality, we get the boundedness of the operator in the right part of (27) in the space L_r, $1 \le r \le \infty$. Now let us evaluate the limit of the right side of (27), when $\varepsilon \to 0+$ in norm of the space L_r. We begin by representing the function $g_\varepsilon(\tau)$ as follows

$$g_\varepsilon(\tau) = \frac{1}{2\pi} \int_{-\infty}^\infty \frac{g(\tau - \varepsilon t)}{t^2+1} h(\tau, \tau - \varepsilon t, \varepsilon) dt - \frac{1}{2\pi} \int_{-\infty}^\infty \frac{g(\varepsilon t - \tau)}{t^2+1} h(\tau, \varepsilon t - \tau, \varepsilon) dt = g_{1\varepsilon}(\tau) - g_{2\varepsilon}(\tau), \tag{30}$$

where

$$h(\tau, t, \varepsilon) = \frac{2^\varepsilon \sinh(\alpha\,\tau) \sinh((\pi-\varepsilon)t)}{\pi \Gamma(\varepsilon+1) \tau \sinh(\alpha\,\tau)} \times \left| \Gamma\left(\frac{\varepsilon + i(t+\tau)}{2} + 1\right) \Gamma\left(\frac{\varepsilon + i(t-\tau)}{2} + 1\right) \right|^2. \tag{31}$$

From the previous discussion, we conclude that the function $h(\tau,t,\varepsilon)$ is bounded uniformly for all parameters $r>0$, $t\in R$, $\varepsilon\in(0,\pi)$. Moreover, from the reduction and supplement formulae for gamma-function the following limit relation takes place

$$\lim_{\varepsilon\to 0+} h(\tau,\tau-\varepsilon t,\varepsilon)=\lim_{\varepsilon\to 0+} h(\tau,\varepsilon t-\tau,\varepsilon)=1. \quad (32)$$

Hence we obtain the following estimates for norms of the functions $g_{i\varepsilon}(\tau)$, $i=1,2$ in the space $L_r(R_+)$

$$\| g_{1\varepsilon}(\tau)-\frac{g(\tau)}{2}\|_{L_r(R_+)} \quad (33)$$
$$<\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{t^2+1}$$
$$\| g(\tau-\varepsilon t)h(\tau,\tau-\varepsilon t,\varepsilon)-g(\tau)\|_{L_r(R_+)}dt\to 0,\ \varepsilon\to 0+$$

$$\| g_{2\varepsilon}(\tau)-\frac{g(-\tau)}{2}\|_{L_r(R_+)} \quad (34)$$
$$<\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{t^2+1}$$
$$\times\| g(\varepsilon t-\tau)h(\tau,\varepsilon-\tau,\varepsilon)-g(-\tau)\|_{L_r(R_+)}dt\to 0,\ \varepsilon\to 0+$$

Combining (33)-(34) with representation (30) and applying the Minkowski inequality for the norm of a sum of functions, we find that

$$\| g_\varepsilon(\tau)-g(\tau)\|_{L_r(R_+)}=\| g_\varepsilon(\tau)-\frac{g(\tau)-g(-\tau)}{2}\|_{L_r(R_+)}$$
$$\le\| g_{1\varepsilon}(\tau)-\frac{g(\tau)}{2}\|_{L_r(R_+)}$$
$$+\| g_{2\varepsilon}(\tau)-\frac{g(-\tau)}{2}\|_{L_r(R_+)}\to 0,\ \varepsilon\to 0+. \quad (35)$$

But, on the other side, since the operator $K L_\alpha$ is bounded on L_p-space, where $1\le p\le\infty$, there exists the following limit in L_p-norm

$$Lim_{\varepsilon\to 0+}[K L_\alpha(I_\varepsilon^\alpha g)]=$$
$$[K L_\alpha\, Lim_{\varepsilon\to 0+}(I_\varepsilon^\alpha g)]=[K L_\alpha f], \quad (36)$$

where $f=I^\alpha g\in L_p$. Since the operator $[K L_\alpha(I_\varepsilon^\alpha g)]$ converges in the norm L_r too, then the limit functions must coincide almost everywhere on R_+. Thus, from equality (36) we obtain (25).

The Lebedev-Skalskaya transforms

Now we construct the L_p-theory for Lebedev-Skalskaya transform pairs, which were introduced in [3] following the corresponding results for the Kontorovich-Lebedev transform in the section. 2. First, we use the integral representation for the Macdonald function as follows,

$$K_\nu(x)=\int_0^\infty e^{-x\cosh(u)}\cosh(\nu u)du \quad (37)$$

and further, are have

$$ReK_{1/2+i\tau}(x)=\int_0^\infty e^{-x\cosh(u)}\cosh(u/2)\cos(\tau u)du,\ x>0, \quad (38)$$

$$ImK_{1/2+i\tau}(x)=\int_0^\infty e^{-x\cosh(u)}\sinh(u/2)\sin(\tau u)du,\ x>0. \quad (39)$$

where the left parts of the equalities (38)-(39) are defined as

$$\begin{Bmatrix}Re\\Im\end{Bmatrix}K_{1/2+i\tau}(x)=\frac{K_{1/2+i\tau}(x)\pm K_{1/2-i\tau}(x)}{2\begin{Bmatrix}1\\i\end{Bmatrix}} \quad (40)$$

For further considerations, the following lemma concerning the uniform estimation by the variables $\tau>0$, $x>0$ of the kernels (40) will be useful.

Lemma 1

For arbitrary $\delta\in[0,\pi/2)$ and for all $\tau>0$, $x>0$ the estimate

$$\left|\begin{Bmatrix}Re\\Im\end{Bmatrix}K_{1/2+i\tau}(x)\right|\le(\pi/2)^{1/2}\times$$
$$(\cos(\delta))^{1/2}e^{-\delta\tau-x\cos(\delta)}x^{-1/2} \quad (41)$$

holds.

Proof. By analytic properties of the integrand in (38)-(39), we can rewrite these representations as

$$ReK_{1/2+i\tau}(x)=\frac{1}{2}\int_{\delta i-\infty}^{\delta i+\infty}e^{-x\cosh(\beta)}e^{i\tau\beta}\cosh(\beta/2)d\beta,\ x>0, \quad (42)$$

$$ImK_{1/2+i\tau}(x)=\frac{1}{2}\int_{\delta i-\infty}^{\delta i+\infty}e^{-x\cosh(\beta)}e^{i\tau\beta}\sinh(\beta/2)d\beta,\ x>0. \quad (43)$$

It is not difficult to obtain

$$\left|\begin{Bmatrix}\mathrm{Re}\\ \mathrm{Im}\end{Bmatrix} K_{1/2+i\tau}(x)\right| \le e^{-\delta\tau}$$

$$\frac{1}{2}\int_{-\infty}^{+\infty} e^{-x\cos(\delta)\cosh(u)}\cosh(u/2)du$$

$$= e^{\delta\tau} K_{1/2}(x\cos(\delta)) = (\pi/2)^{1/2} e^{-\delta\tau - x\cos(\delta)} \times (x\cos(\delta))^{-1/2}. \quad (44)$$

Let us consider the Lebedev-Skalskaya transforms of type

$$[LS_\alpha^{\{\mathrm{Re},\mathrm{Im}\}} f](\tau) = \cos(\alpha\tau)\int_0^\infty \begin{Bmatrix}\mathrm{Re}\\ \mathrm{Im}\end{Bmatrix} K_{1/2+i\tau}(y) y^{1/2} f(y)dy. \quad (45)$$

where $0 < \alpha < \pi/2$, $f(y) \in L_p(R_+)$, $1 \le p \le \infty$. As it is evident from the Hölder inequality and from the asymptotic behaviour of the Macdonald function (8)-(9), the integral (45) absolutely converges for any $p \ge 1$. Let us consider the following similar spaces of functions $g(\tau)$ which can be represented by the Lebedev-Skalskaya transforms, where the respective function $f(y)$ belongs to $L_p(R_+)$

$$LS_\alpha^{\{\mathrm{Re},\mathrm{Im}\}}(L_p) = \{g : g(\tau) = [LS_\alpha^{\{\mathrm{Re},\mathrm{Im}\}} f](\tau), f \in L_p(R_+)\}, \quad 0 < \alpha < \pi/2,\ p \ge 1. \quad (46)$$

Making use of Lemma 1 and applying the general Minkowski inequality to the integral (45), we obtain that the operators $[LS_\alpha^{\{\mathrm{Re},\mathrm{Im}\}} f]$ are bounded for mappings from the space $L_p(R_+)$, $1 \le p \le \infty$, into the space $L_q(R_+)$, $1 \le q \le \infty$. Indeed, we have

$$\| [LS_\alpha^{\{\mathrm{Re},\mathrm{Im}\}} f] \|_{L_q(R_+)} \le \int_0^\infty |f(y)| \times \left(\int_0^\infty \cosh^q(\alpha\tau) \left|\begin{Bmatrix}\mathrm{Re}\\ \mathrm{Im}\end{Bmatrix} K_{1/2+i\tau}(y) y^{1/2}\right|^q d\tau\right)^{1/q} dy$$

$$\le C_1 \int_0^\infty |f(y)| e^{-y\cos(\delta)} \left(\int_0^\infty e^{q\tau(\alpha-\delta)}d\tau\right)^{1/q} dy \le C_\delta \| f \|_{L_p}, \quad 1 \le p \le \infty, \quad (47)$$

where C_1, C_δ are positive constants and we choose $\delta > \alpha$.

In order to describe the spaces $LS_\alpha^{\{\mathrm{Re},\mathrm{Im}\}}(L_p)$ let us consider the following operators

$$\left(\begin{Bmatrix}\mathrm{Re}\\ \mathrm{Im}\end{Bmatrix}_\varepsilon^\alpha g\right)(x) = x^{\varepsilon-1/2}\frac{4}{\pi^2}\int_0^\infty \frac{\cosh((\pi-2\varepsilon)\tau)}{\cosh(\alpha\tau)} \begin{Bmatrix}\mathrm{Re}\\ \mathrm{Im}\end{Bmatrix} K_{1/2+i\tau}(x) g(\tau)d\tau, \quad (48)$$

where $\varepsilon \in (0,\pi/2)$.

Theorem 4

On the functions $g(\tau) = [LS_\alpha^{\{\mathrm{Re},\mathrm{Im}\}} f](\tau)$, which are represented by the Lebedev-Skalskaya transforms (45) with the density $f(y) \in L_p(R_+)$, $1 \le p \le \infty$, the operators (48) have the following form

$$\left(\begin{Bmatrix}\mathrm{Re}\\ \mathrm{Im}\end{Bmatrix}_\varepsilon^\alpha g\right)(x) = \{\pm\} x^{\varepsilon-1/2}\frac{\sin(\varepsilon)}{\pi} \times \int_0^\infty K_0\left(\sqrt{x^2+y^2-2xy\cos(2\varepsilon)}\right)\sqrt{y}f(y)dy + x^{\varepsilon-1/2}\frac{\sin(\varepsilon)}{\pi}$$

$$\times \int_0^\infty \frac{(x+y)\sqrt{y}K_1\left(\sqrt{x^2+y^2-2xy\cos(2\varepsilon)}\right)}{\sqrt{x^2+y^2-2xy\cos(2\varepsilon)}} f(y)dy,\ x > 0. \quad (49)$$

Proof. Substituting the value of $g(\tau)$ as a Lebedev-Skalskaya transforms (45) in formula (48) and changing order of integration in the absolutely convergent iterated integral, we use the following integral 2.16.55.2 from [4], Vol. 2

$$\int_0^\infty \cos(a\tau)\begin{Bmatrix}\mathrm{Re}\\ \mathrm{Im}\end{Bmatrix} K_{1/2+i\tau}(b) \times \begin{Bmatrix}\mathrm{Re}\\ \mathrm{Im}\end{Bmatrix} K_{1/2+i\tau}(c)d\tau = \frac{\pi}{4}\cosh(a/2) \times \left(\pm K_0(z) + \frac{b+c}{z}K_1(z)\right),\ z = (b^2+c^2+2bc\cosh(a))^{1/2} \quad (50)$$

and the Fubini theorem, to get representation (49).

The analogue of theorem 2 gives the following:

Theorem 5

Let $g(\tau) = [LS_\alpha^{\{\substack{Re\\Im}\}} f](\tau)$, $f(y) \in L_p(R_+)$, $1 \le p < \infty$. *Then*

$$f(x) = \left(\begin{Bmatrix} \mathrm{Re} \\ \mathrm{Im} \end{Bmatrix}^{\alpha} g \right)(x) \quad (51)$$

where $\left(\begin{Bmatrix} \mathrm{Re} \\ \mathrm{Im} \end{Bmatrix}^{\alpha} g \right)(x)$ *is understood as*

$$\left(\begin{Bmatrix} \mathrm{Re} \\ \mathrm{Im} \end{Bmatrix}^{\alpha} g \right)(x) = \lim_{\varepsilon \to 0+} \left(\begin{Bmatrix} \mathrm{Re} \\ \mathrm{Im} \end{Bmatrix}_{\varepsilon}^{\alpha} g \right)(x), \; x > 0 \quad (52)$$

and the limit in equality (52) *is understood in the norm in* L_p. *Moreover, the limit in* (52) *exists almost everywhere on* R_+.

Proof. The proof of this theorem follows respective treatment of integrals (49). Indeed, after replacement of variable $y = x(\cos(2\varepsilon) + t\sin(2\varepsilon))$, we get the equality

$$\left(\begin{Bmatrix} \mathrm{Re} \\ \mathrm{Im} \end{Bmatrix}_{\varepsilon}^{\alpha} g \right)(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{t^2+1} f(x(\cos(2\varepsilon) + t\sin(2\varepsilon))$$

$$\times (\cos(2\varepsilon) + t\sin(2\varepsilon)) \hat{R}(x,t,\varepsilon)\, dt, \; \varepsilon \in (0,\pi/2), \quad (53)$$

where

$$\hat{R}(x,t,\varepsilon) = \pm x^{\varepsilon+1} \sin(\varepsilon)\sin(2\varepsilon)(\cos(2\varepsilon) + t\sin(2\varepsilon))^{-1/2}$$

$$\times (t^2+1) K_0(x\sin(2\varepsilon)(t^2+1)^{1/2})$$

$$+ \frac{x^{\varepsilon+1}(1 + \cos(2\varepsilon) + t\sin(2\varepsilon))}{2\cos(\varepsilon)\sqrt{\cos(2\varepsilon) + t\sin(2\varepsilon)}}$$

$$\times \sin(2\varepsilon)(t^2+1)^{1/2} K_1(x\sin(2\varepsilon)(t^2+1)^{1/2}), \; t \ge -ctg(\varepsilon), \quad (54)$$

$$\hat{R}(x,t,\varepsilon) = 0, \; t < -ctg(\varepsilon) \; . \quad (55)$$

From the asymptotic behaviour of Macdonald functions $K_0(z)$, $K_1(z)$, for any $t \in R$, $x \in R_+$ and $\varepsilon \in (0,\pi)$ $|\hat{R}(x,t,\varepsilon)| < C$ where C is a positive constant,

$$\lim_{\varepsilon \to 0+} \hat{R}(x,t,\varepsilon) = 1.$$

Further, we use the approximation properties of the Poisson kernel $P(t) = \frac{1}{\pi}\frac{1}{t^2+1}$ as in section 2 and we estimate the following L_p-norm applying the general Minkowski inequality, namely

$$\left\| \left(\begin{Bmatrix} \mathrm{Re} \\ \mathrm{Im} \end{Bmatrix}_{\varepsilon}^{\alpha} g \right) - f \right\|_{L_p(R_+)} \le \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{t^2+1}$$

$$\| f(x(\cos(2\varepsilon) + t\sin(2\varepsilon))(\cos(2\varepsilon)$$

$$+ t\sin(2\varepsilon)) \hat{R}(x,t,\varepsilon) - f(x) \|_{L_p(R_+)} dt \to 0, \; \varepsilon \to 0+. \quad (56)$$

Indeed, due to the estimate

$$\left\| \left(\begin{Bmatrix} \mathrm{Re} \\ \mathrm{Im} \end{Bmatrix}_{\varepsilon}^{\alpha} g \right) \right\|_{L_p(R_+)} < C \| f \|_{L_p(R_+)}, \; 1 \le p < \infty, \quad (57)$$

where C is a positive absolute constant, from the Lebesgue theorem and the continuity of L_p-norms we prove equality (52). The existence of the limit almost everywhere on R_+ follows from the radial property of Poisson kernel $P(t) = P(|t|) \in L_1(R_+)$. Theorem 5 is proved.

From estimate (57), we have

$$\left\| \left(\begin{Bmatrix} \mathrm{Re} \\ \mathrm{Im} \end{Bmatrix}_{\varepsilon}^{\alpha} g \right) \right\|_{L_p(R_+)} < C \left\| \left(\begin{Bmatrix} \mathrm{Re} \\ \mathrm{Im} \end{Bmatrix}^{\alpha} g \right) \right\|_{L_p(R_+)},$$

$$g \in LS_\alpha^{\{\substack{Re\\Im}\}}(L_p), \; 1 \le p < \infty. \quad (58)$$

From theorem 5, it follows that $[LS_\alpha^{\{\substack{Re\\Im}\}} f](\tau) \equiv 0$, $f(y) \in L_p(R_+)$, $1 \le p < \infty$, iff $f(y) \equiv 0$. Thus, in the space $LS_\alpha^{\left\{\begin{smallmatrix}Re\\Im\end{smallmatrix}\right\}}(L_p)$ we can introduce a norm by the equality

$$\|g\|_{LS_\alpha^{\left\{\begin{smallmatrix}Re\\Im\end{smallmatrix}\right\}}(L_p)} = \|f\|_{L_p},\ g = [LS_\alpha^{\left\{\begin{smallmatrix}Re\\Im\end{smallmatrix}\right\}} f] \qquad (59)$$

As it is evident, the spaces $LS_\alpha^{\left\{\begin{smallmatrix}Re\\Im\end{smallmatrix}\right\}}(L_p)$ are Banach with the norms (59) and they are isometric to L_p.

Let us prove the following descriptions of the spaces $LS_\alpha^{\left\{\begin{smallmatrix}Re\\Im\end{smallmatrix}\right\}}(L_p)$ in term of the operators (48).

Theorem 6

An arbitrary function $g(\tau)$ is defined on R_+ and even in the $\Re$- and odd in the $\Im$-cases, when continued to R, belongs to the spaces $LS_\alpha^{\left\{\begin{smallmatrix}Re\\Im\end{smallmatrix}\right\}}(L_p)$, $1 \le p < \infty$, if and only if $g(\tau) \in L_r(R_+)$, $1 \le r \le \infty$ and the following conditions hold

$$l.i.m._{\varepsilon\to 0+}\left(\left\{\begin{matrix}Re\\Im\end{matrix}\right\}_\varepsilon^\alpha g\right) \in L_p(R_+). \qquad (60)$$

Proof. The necessity of condition (60) follows from previous theorem 5 and from estimate (46). Let us prove the sufficiency. Let $g(\tau) \in L_r(R_+)$, $g(\tau) = \pm g(-\tau)$ and condition (60) hold. We show that in this case there is a function $f \in L_p$ such that the equality

$$g = [LS_\alpha^{\left\{\begin{smallmatrix}Re\\Im\end{smallmatrix}\right\}} f] \qquad (61)$$

takes place. From inequality (58), we conclude that $\left(\left\{\begin{matrix}Re\\Im\end{matrix}\right\}_\varepsilon^\alpha g\right) \in L_p$ for each $\varepsilon \in (0,\pi)$ and we can evaluate the following composition

$$[LS_\alpha^{\left\{\begin{smallmatrix}Re\\Im\end{smallmatrix}\right\}}\left(\left\{\begin{matrix}Re\\Im\end{matrix}\right\}_\varepsilon^\alpha g\right)](\tau) =$$

$$\cosh(\alpha\tau)\int_0^\infty \left\{\begin{matrix}Re\\Im\end{matrix}\right\} K_{1/2+i\tau}(y) y^{1/2}\left(\left\{\begin{matrix}Re\\Im\end{matrix}\right\}_\varepsilon^\alpha g\right)(y)dy. \quad (62)$$

At least for the set of smooth functions with compact support on R_+, which is dense in L_r, we substitute the operator (48) in equality (62) and change order of integration. We need the values of the inner integrals, namely

$$K(\tau,t) = \int_0^\infty y^\varepsilon \left\{\begin{matrix}Re\\Im\end{matrix}\right\} K_{1/2+i\tau}(y) \left\{\begin{matrix}Re\\Im\end{matrix}\right\} K_{1/2+it}(y)dy =$$

$$(\pm)\frac{2^{\varepsilon-2}}{\Gamma(\varepsilon)} \times \frac{\left|\Gamma\left(\frac{\varepsilon+i(t+\tau)}{2}+1\right)\Gamma\left(\frac{\varepsilon+1+i(\tau-t)}{2}\right)\right|^2}{\varepsilon^2+(\tau+t)^2}$$

$$+\frac{2^{\varepsilon-2}}{\Gamma(\varepsilon)}\frac{\left|\Gamma\left(\frac{\varepsilon+i(\tau-t)}{2}+1\right)\Gamma\left(\frac{\varepsilon+1+i(\tau+t)}{2}+1\right)\right|^2}{\varepsilon^2+(\tau-t)^2}, \qquad (63)$$

which were obtained using formula 2.16.33.2 from [4], Vol. 2. Thus we represent equality (62) as follows

$$[LS_\alpha^{\left\{\begin{smallmatrix}\Re\\\Im\end{smallmatrix}\right\}}\left(\left\{\begin{matrix}\Re\\\Im\end{matrix}\right\}_\varepsilon^\alpha g\right)](\tau) = g_\varepsilon(\tau) =$$

$$\cosh(\alpha\tau)\frac{4}{\pi^2}\int_0^\infty \frac{\cosh((\pi-2\varepsilon)t)}{\cosh(\alpha\tau)} K(\tau,t)g(t)dt \quad (64)$$

for $\varepsilon \in (0,\pi/2)$. In order to prove equality (64) for all $g \in L_r$, we must prove the boundedness of the operator in the right side of (64). But the kernel of the integrand in (64) is equal to

$$O(e^{(\pi/2-\alpha)(t-\tau)-\pi/2|t-\tau|-2\varepsilon t}),\ (t,\tau) \in R_+\times R_+,$$
$$\alpha \in (0,\pi/2),\ \varepsilon \in (0,\pi/2). \qquad (65)$$

Now repeating the previous discussions in section 2, we prove the boundedness of the operator in the right side of (64) in the space L_r, $1 \le r \le \infty$. Hence representing the function $g_\varepsilon(\tau)$ as

$$g_\varepsilon(\tau) = \pm\frac{1}{2\pi}\int_{-\infty}^\infty \frac{g(\varepsilon t-\tau)}{t^2+1}\hat{h}(\tau,\varepsilon t-\tau,\varepsilon)dt$$

$$+\frac{1}{2\pi}\int_{-\infty}^\infty \frac{g(\tau-\varepsilon t)}{t^2+1}\hat{h}(\tau,\varepsilon t-\tau,\varepsilon)dt = g_{1\varepsilon}(\tau)+g_{2\varepsilon}(\tau), \quad (66)$$

where

$$\hat{h}(\tau,t,\varepsilon) = \frac{2^{\varepsilon}\cosh(\alpha\tau)\cosh((\pi-2\varepsilon)t)}{\pi\Gamma(\varepsilon+1)\tau\cosh(\alpha t)} \times \left|\Gamma\left(\frac{\varepsilon+i(t+\tau)}{2}+1\right)\Gamma\left(\frac{\varepsilon+i(\tau-t)+1}{2}\right)\right|^2.$$

From the previous discussions we conclude that the function $\hat{h}(\tau,t,\varepsilon)$ is bounded uniformly for all parameters $\tau>0$, $t\in R$, $\varepsilon\in(0,\pi/2)$. Moreover, from the reduction and supplement formulae for gamma-function the following limit relation takes place

$$\lim_{\varepsilon\to 0+}\hat{h}(\tau,\varepsilon t-\tau,\varepsilon) = 1. \tag{67}$$

Hence as in theorem 3 we obtain the followingt estimates for the norms of the functions $g_{i\varepsilon}(\tau)$, $i=1,2$ in the space $L_r(R_+)$

$$\| g_{1\varepsilon}(\tau) \mp \frac{g(-\tau)}{2} \|_{L_r(R_+)} < \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{t^2+1} \times \|\pm g(\varepsilon t-\tau)\hat{h}(\tau,\varepsilon t-\tau,\varepsilon) \mp g(-\tau)\|_{L_r(R_+)}dt \to 0,\ \varepsilon\to 0+ \tag{68}$$

$$\| g_{2\varepsilon}(\tau) - \frac{g(\tau)}{2} \|_{L_r(R_+)} < \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{t^2+1} \times \| g(\tau-\varepsilon t)\hat{h}(\tau,\varepsilon t-\tau,\varepsilon) - g(\tau)\|_{L_r(R_+)}dt \to 0,\ \varepsilon\to 0+ \tag{69}$$

Combining (68)-(69) with the representation (66) and applying the usual Minkowski inequality for the norm of sum of functions, we obtain

$$\| g_{\varepsilon}(\tau) - g(\tau) \|_{L_r(R_+)} = \| g_{\varepsilon}(\tau) - \frac{g(\tau)\pm g(-\tau)}{2} \|_{L_r(R_+)} \le \| g_{1\varepsilon}(\tau) \mp \frac{g(-\tau)}{2} \|_{L_r(R_+)} + \| g_{2\varepsilon}(\tau) - \frac{g(\tau)}{2} \|_{L_r(R_+)} \to 0,\ \varepsilon\to 0+. \tag{70}$$

But, on the other hand, since the operator $LS_{\alpha}^{\left\{\begin{smallmatrix}Re\\Im\end{smallmatrix}\right.}$ is bounded in L_p-space, where $1\le p<\infty$, there exists the following limit in L_p-norm

$$l.i.m._{\varepsilon\to 0+}[LS_{\alpha}^{\left\{\begin{smallmatrix}Re\\Im\end{smallmatrix}\right.}\left(\begin{Bmatrix}Re\\Im\end{Bmatrix}_{\varepsilon}^{\alpha} g\right)] = [LS_{\alpha}^{\left\{\begin{smallmatrix}Re\\Im\end{smallmatrix}\right.}\ l.i.m._{\varepsilon\to 0+}\left(\begin{Bmatrix}Re\\Im\end{Bmatrix}_{\varepsilon}^{\alpha} g\right)] = [LS_{\alpha}^{\left\{\begin{smallmatrix}Re\\Im\end{smallmatrix}\right.} f], \tag{71}$$

where $f=\begin{Bmatrix}Re\\Im\end{Bmatrix}_{\varepsilon}^{\alpha} g\in L_p$. Since the operator $[LS_{\alpha}^{\left\{\begin{smallmatrix}Re\\Im\end{smallmatrix}\right.}\left(\begin{Bmatrix}Re\\Im\end{Bmatrix}_{\varepsilon}^{\alpha} g\right)]$ converges in the norm L_r, too, the limit functions must coincide almost everywhere on R_+. Thus from equality (71) we obtain (60).

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