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Expressions of Legendre polynomials through Bernoulli polynomials

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Abstract

A formula for expanding Legendre polynomials in Bernoulli polynomials is considered. The relationship is established by using a formula of finite summation, obtained by applying the discrete orthogonal relation of the modified Lommel polynomials.

Key words: Legendre polynomials, Bernoulli polynomials.

Expresión de los polinomios de Legendre a través de los polinomios de Bernoulli

Resumen

En este trabajo se presenta una relación entre los polinomios de Legendre y los polinomios de Bernoulli. Concretamente, todos los polinomios de Legendre son expresados mediante polinomios de Bernoulli. Para establecer esta relación, se ha utilizado una fórmula de suma finita, cuya prueba se obtiene aplicando la ortogonalidad de los polinomnios modificados de Lommel.

Palabras claves: Polinomios de Legendre, polinomios de Bernoulli.

Introduction

The main aim of this work is to obtain some expansion formulas of Legendre polynomials in Bernoulli polynomials.

Legendre polynomials are defined by

$$P_n(x) = \frac{1}{n!2^n} \frac{d^n(x^2-1)^n}{dx^n}, \quad n \geq 0,$$

and satisfy the orthogonal relation

$$\int_{-1}^{1} P_n(x)P_m(x)dx = \frac{2}{2n+1}\delta_{mn}, \qquad (1)$$

([3], Chapter 3, 3.12.8 and 3.12.10), where

$$\delta_{mn} = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases}$$

is the "Kronecker delta". By a linear change of x the relation (1) becomes

$$\int_{0}^{1} P_n(2x-1)P_m(2x-1)dx = \frac{1}{2n+1}\delta_{mn}. \qquad (2)$$

Bernoulli numbers B_n, $n \geq 0$, are defined by

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \quad |z| < 2\pi,$$

whereas Bernoulli polynomials $B_n(x)$, $n \geq 0$, are defined by

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!}, \quad |z| < 2\pi,$$

([3], Chapter 1, $ 1.13).

Let us recall some properties of Bernoulli numbers and polynomials that will be used in the next sections

$$B_1 = -1/2; \quad B_{2m+1} = 0, \quad m > 0; \quad B_n(0) = B_n, \tag{3}$$

$$\int_0^1 B_m(x) B_n(x) dx = (-1)^{m+1} \frac{m! n!}{(m+n)!} B_{m+n}, \quad m, n \neq 0, \tag{4}$$

$$\int_0^1 B_{2m+1}(x) dx = 0, \quad m \geq 0, \tag{5}$$

([3], Chapter 1, $ 1.13, formulae (4),(17),(19); [5], Chapter 2, p. 55, $ 2.4.1, formula (1), $ 2.4.2, formula (2)).

The modified Lommel polynomials ([1], [2]) are defined by

$$h_{n,\upsilon}(x) = (\upsilon)_n (2x)^n \; . \; {}_2F_3(-n/2,(1-n)/2;\upsilon,-n,1-\upsilon-n;-1/x^2), \tag{6}$$

where $(\upsilon)_n = \dfrac{\Gamma(\upsilon+n)}{\Gamma(\upsilon)}$ is the Pchhammer symbol and ${}_2F_3(a_1,a_2;b_1,b_2,b_3;z)$ is a generalized hypergeometric function ([3],Chapter 4).

Let $\{j_{\upsilon,n}\}$, $n = \pm 1, \pm 2, \ldots$, be the nonvanishing zeros of theBessel function $J_\upsilon(x)$ ([4], Chapter 7) ordered by

$$\cdots < j_{\upsilon,-2} < j_{\upsilon,-1} < 0 < j_{\upsilon,1} < j_{\upsilon,2} < \cdots .$$

The modified Lommel polynomials $h_{n,\upsilon}(x)$ satisfy the discrete orthogonal relationship

$$\sum_{k=-\infty}^{\infty}{}' h_{n,\upsilon}\left(\frac{1}{j_{\upsilon-1,k}}\right) h_{m,\upsilon}\left(\frac{1}{j_{\upsilon-1,k}}\right) \frac{1}{j^2_{\upsilon-1,k}} = \frac{\delta_{mn}}{2(\upsilon+n)}, \tag{7}$$

where the dash in the sum indicates that the term with index $k = 0$ is omitted, and the recurrence formula ([2])

$$h_{n+1,\upsilon}(x) = 2x(n+\upsilon) h_{n,\upsilon}(x) - h_{n-1,\upsilon}(x). \tag{8}$$

Incidentally, we notice that formula (7) appeared in [1,2] with the incorrect right-hand side.

Formula of finite summation

The following formula of finite summation containing Bernoulli numbers has been published in [5], Chapter 5, 5.1.1.6 in incorrect form. Since it plays an important role in the next section and we could not find its proof in literature, we shall formulate it as a lemma.

Lemma

For $\sigma = 0$ or 1 the formula

$$\sum_{l=0}^{n} \frac{(2n+2l+2\sigma+1)! B_{2m+2l+2\sigma+2}}{(2l+\sigma)!(2n-2l+1)!(2m+2l+2\sigma+2)!} = (-1)^\sigma \frac{(2n+\sigma+1)!}{(4n+2\sigma+3)! 2} \delta_{mn}, \quad (0 \leq m \leq n) \tag{9}$$

is valid.

Proof. We shall prove (9) by using the discrete orthogonal relation (7) of the modified Lommel polynomials when $\upsilon = 3/2$.

First we consider the case $\sigma = 0$. From (6) it is easy to see that the modified Lommel polynomial $h_{n,\upsilon}(x)$ is an even or an odd polynomial according as n is even or odd, i.e. $h_{n,\upsilon}(-x) = (-1)^n h_{n,\upsilon}(x)$. Therefore,

$$\upsilon_n(x) = h_{2n,3/2}\left(\frac{2\sqrt{x}}{\pi}\right), \quad n = 0, 1, \ldots, \tag{10}$$

is a polynomial of precise degree n in variable x. Since ([4], Chapter 7, $ 7.11, formula (14), p. 79)

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x,$$

then

$$j_{1/2,k} = k\pi, \quad k = \pm 1, \pm 2, \ldots$$

Therefore, the discrete orthogonal relation (7) for the sequence of polynomials $\{\upsilon_n(x)\}$ becomes

$$\sum_{k=1}^{\infty} \upsilon_n\left(\frac{1}{4k^2}\right) \upsilon_m\left(\frac{1}{4k^2}\right) \cdot \frac{1}{k^2} = \frac{\pi^2}{2(3+4n)} \delta_{mn}, \quad 0 \leq m \leq n \tag{11}$$

Now we establish the explicit expression of the polynomials $\upsilon_n(x)$. From(6) and (10) we have

$$\upsilon_n(x)=(3/2)_{2n}\frac{4^{2n}x^n}{\pi^{2n}}$$

$$_2F_3(-n,\ 1/2-n;3/2,-2n,-1/2-2n;-\frac{\pi^2}{4x})$$

$$=\frac{(4n+1)!!2^{2n}x^n}{\pi^{2n}}$$

$$\sum_{k=0}^{n}\frac{(-n)_k(1/2-n)_k}{(1)_k(3/2)_k(-1/2-2n)_k(-2n)_k}(-\frac{\pi^2}{4x})^k,$$

where $k!! = k(k-2)(k-4)\dots(k-2[\frac{k}{2}]+2)$. Using the formula

$$4^k(a)_k(a+1/2)_k=(2a)_{2k},$$

we obtain

$$\upsilon_n(x)=\frac{(4n+1)!!2^{2n}x^n}{\pi^{2n}}\sum_{k=0}^{n}\frac{(-2n)_{2k}}{(2)_{2k}(-4n-1)_{2k}}(-\frac{\pi^2}{x})^k$$

$$=\frac{(4n+1)!!2^{2n}x^n}{\pi^{2n}}\cdot$$

$$\sum_{k=0}^{n}\frac{(2n-2k+1)_{2k}}{(2k+1)!(4n-2k+2)_{2k}}(-\frac{\pi^2}{x})^k.$$

By changing now n-k by k we get

$$\upsilon_n(x)=(-1)^n2^{2n}(4n+1)!!\ \cdot$$

$$\sum_{k=0}^{n}\frac{(2k+1)_{2n-2k}}{(2n-2k+1)!(2n+2k+2)_{2n-2k}}(-\frac{x}{\pi^2})^k$$

$$=(-1)^n2^{2n}(4n+1)!!\ \cdot$$

$$\sum_{k=0}^{n}\frac{(2n)!(2n+2k+1)!}{(2k)!(2n-2k+1)!(4n+1)!}(-\frac{x}{\pi^2})^k$$

$$=(-1)^n\sum_{k=0}^{n}\frac{(2n+2k+1)!}{(2k)!(2n-2k+1)!}(-\frac{x}{\pi^2})^k. \qquad (12)$$

Now the orthogonal relation (11) becomes

$$\sum_{k=1}^{\infty}\upsilon_n(\frac{1}{4k^2})\cdot(\frac{1}{4k^2})^m\cdot\frac{1}{k^2}=\frac{(2n+1)!\pi^{2n+2}}{(4n+3)!}\delta_{mn},$$

$$0\le m\le n. \qquad (13)$$

Putting the explicit representation (12) of $\upsilon_n(x)$ into (13) we get

$$\sum_{k=1}^{\infty}\left[\sum_{l=0}^{n}\frac{(-1)^{n+l}(2n+2l+1)!}{(2l)!(2n-2l+1)!\pi^{2l}}(\frac{1}{4k^2})^l\right](\frac{1}{4k^2})^m\cdot\frac{1}{k^2}=$$

$$\frac{(2n+1)!\pi^{2n+2}}{(4n+3)!}\delta_{mn},\quad 0\le m\le n. \qquad (14)$$

By changing the order of summation in (14) we obtain

$$\sum_{l=0}^{n}\frac{(-1)^{n+l}(2n+2l+1)!}{4^{l+m}(2l)!(2n-2l+1)!\pi^{2l}}\sum_{k=1}^{\infty}\frac{1}{k^{2m+2l+2}}=$$

$$\frac{(2n+1)!\pi^{2n+2}}{(4n+3)!}\delta_{mn},\quad 0\le m\le n.$$

Applying the formula ([3],Chapter 1, \$ 1.13, formula (16), p. 38)

$$\sum_{k=1}^{\infty}\frac{1}{k^{2m+2l+2}}=\frac{(-1)^{m+l}(2\pi)^{2m+2l+2}}{(2m+2l+2)!2}B_{2m+2l+2}, \qquad (15)$$

we get

$$\sum_{l=0}^{n}\frac{(2n+2l+1)!B_{2m+2l+2}}{(2l)!(2n-2l+1)!(2m+2l+2)!}=$$

$$\frac{(2n+1)!}{(4n+3)!2}\delta_{mn},\quad 0\le m\le n. \qquad (16)$$

Hence formula (9) is proved in case $\sigma=0$.

We consider now the case $\sigma=1$. Let

$$t_n(x)=x^{-1}[\upsilon_{n+1}(x)+\upsilon_n(x)]. \qquad (17)$$

Since $\upsilon_n(0)=(-1)^n$, it is easy to see that $t_n(x)$ is a polynomial of precise degree n. We have

$$\sum_{k=1}^{\infty}t_n(\frac{1}{4k^2})\cdot(\frac{1}{4k^2})^m\cdot\frac{1}{k^4}= \qquad (18)$$

$$= \sum_{k=1}^{\infty} \upsilon_{n+1}(\frac{1}{4k^2}) \cdot (\frac{1}{4k^2})^m \cdot \frac{4}{k^2}$$

$$+ \sum_{k=1}^{\infty} \upsilon_n(\frac{1}{4k^2}) \cdot (\frac{1}{4k^2})^m \cdot \frac{4}{k^2}$$

$$= 4 \sum_{k=1}^{\infty} \upsilon_n(\frac{1}{4k^2}) \cdot (\frac{1}{4k^2})^m \cdot \frac{1}{k^2} = \frac{4(2n+1)!\pi^{2n+2}}{(4n+3)!} \delta_{mn},$$

$$0 \le m \le n.$$

After replacing $\upsilon_n(x)$ and $\upsilon_{n+1}(x)$ in (17) by their explicit representation formulas (12) we obtain

$$t_n(x) = \frac{(-1)^n 2(4n+5)}{\pi^2} \cdot \sum_{k=0}^{n} \frac{(2n+2k+3)!}{(2k+1)!(2n-2k+1)!} \left(\frac{-x}{\pi^2}\right)^k . \tag{19}$$

By introducing expression (19) into (18) we get

$$\sum_{k=1}^{\infty} \left[\sum_{l=0}^{n} \frac{(-1)^{n+l}(2n+2l+3)!}{(2l+1)!(2n-2l+1)!\pi^{2l}} (\frac{1}{4k^2})^l \right] \cdot (\frac{1}{4k^2})^m \frac{1}{k^4} = \frac{4(2n+2)!\pi^{2n+4}}{(4n+5)!} \delta_{mn}, \quad 0 \le m \le n. \tag{20}$$

By changing the order of summation in (20) we have

$$\sum_{l=0}^{n} \frac{(-1)^{n+l}(2n+2l+3)!}{4^{l+m}(2l+1)!(2n-2l+1)!\pi^{2l}} \cdot \sum_{k=1}^{\infty} \frac{1}{k^{2m+2l+4}} = \frac{4(2n+2)!\pi^{2n+4}}{(4n+5)!}, \quad 0 \le m \le n. \tag{21}$$

Replacing the infinite sum in (21) by the formula ([3],Chapter 1)

$$\sum_{k=1}^{\infty} \frac{1}{k^{2m+2l+4}} = \frac{(-1)^{m+l+1}(2\pi)^{2m+2l+4}}{(2m+2l+4)!2} B_{2m+2l+4},$$

we obtain

$$\sum_{l=0}^{n} \frac{(2n+2l+3)!B_{2m+2l+4}}{(2l+1)!(2n-2l+1)!(2m+2l+4)!} = -\frac{(2n+2)!}{(4n+5)!2} \delta_{mn}, \quad 0 \le m \le n. \tag{22}$$

Hence, the formula (9) is proved for the case $\sigma = 1$.

Legendre and Bernoulli polynomials

Now we shall use the formula of finite summation (9) to prove the following main theorem.

Theorem

Legendre polynomials are expressed through Bernoulli polynomials by the following formula

$$P_{2n+1+\sigma}(2x-1) = 2 \sum_{k=0}^{n} \frac{(2n+2k+2\sigma+1)!B_{2k+\sigma+1}(x)}{(2k+\sigma)!(2k+\sigma+1)!(2n-2k+1)!}, \tag{23}$$

$$(n \ge 0; \ \sigma = 0 \text{ or } 1).$$

Proof. First, we apply the lemma in case $\sigma = 0$. In formula (16) replacing $B_{2m+2l+2}$ by formula (4) we get

$$\int_0^1 \frac{B_{2m+1}(x)}{(2m+1)!} \sum_{k=0}^{n} \frac{(2n+2k+1)!B_{2k+1}(x)}{(2k)!(2n-2k+1)!(2k+1)!} = \frac{(2n+1)!\delta_{mn}}{(4n+3)!2}, \quad 0 \le m \le n. \tag{24}$$

If we set

$$\sum_{k=0}^{n} \frac{(2n+2k+1)!B_{2k+1}(x)}{(2k)!(2k+1)!(2n-2k+1)!} = P^*_{2n+1}(x), \tag{25}$$

then $P^*_{2n+1}(x)$ is a polynomial of degree $2n+1$ and we obtain

$$\int_0^1 B_{2m+1}(x) P^*_{2n+1}(x)dx = \frac{[(2n+1)!]^2}{(4n+3)!2} \delta_{mn}, \tag{26}$$

$$0 \le m \le n.$$

Using (4) again and noticing that $B_{2m+2k+1} = 0$ for $m > 0$, $k \ge 0$, we have

$$\int_0^1 B_{2m}(x)\, P^*_{2n+1}(x)dx = \int_0^1 B_{2m}(x) \sum_{k=0}^{n} \frac{(2n+2k+1)!B_{2k+1}(x)}{(2k)!(2k+1)!(2n-2k+1)!}\, dx$$

$$=\sum_{k=0}^{n} \frac{(2n+2k+1)!}{(2k)!(2k+1)!(2n-2k+1)!} \int_0^1 B_{2m}(x)\, B_{2k+1}(x)dx$$

$$=\sum_{k=0}^{n} \frac{(-1)^{2m+1}(2n+2k+1)!(2m)!(2k+1)!}{(2k)!(2k+1)!(2n-2k+1)!(2m+2k+1)!} \tag{27}$$

$$B_{2m+2k+1} = 0\ ,\quad m > 0.$$

Finally, applying (5) we obtain

$$\int_0^1 B_0(x)\, P^*_{2n+1}(x)dx = \int_0^1 P^*_{2n+1}(x)dx = \tag{28}$$

$$=\sum_{k=0}^{n} \frac{(2n+2k+1)!}{(2k)!(2k+1)!(2n-2k+1)!} \int_0^1 B_{2k+1}(x)dx = 0.$$

From (26), (27) and (28) we can conclude that the polynomial $P^*_{2n+1}(x)$ is orthogonal to the system $\{B_0(x), B_1(x), \cdots, B_{2n}(x)\}$ with respect to the weight 1 on the interval [0,1]. Since the system of Legendre polynomials $P_n(2x-1)$ is also orthogonal with respect to the weight 1 on [0,1], then there exists a sequence of scalars $\{\alpha_n\}$ such that

$$P^*_{2n+1}(x) = \alpha_n P_{2n+1}(2x-1),\quad n \geq 0. \tag{29}$$

The coefficients α_n can be found exactly. Indeed, putting $x = 0$ into (29) and taking into account formula (3) and the formula

$$P_n(-1) = (-1)^n\ ,$$

we have

$$-\alpha_n = \sum_{k=0}^{n} \frac{(2n+2k+1)!B_{2k+1}}{(2k)!(2k+1)!(2n-2k+1)!} = B_1 = -1/2.$$

Therefore, $\alpha_n = 1/2$. Consequently, formula (29) becomes

$$P_{2n+1}(2x-1) = 2P^*_{2n+1}(x) =$$

$$2\sum_{k=0}^{n} \frac{(2n+2k+1)!B_{2k+1}(x)}{(2k)!(2n-2k+1)!(2k+1)!}, \tag{30}$$

and the proof of the theorem in case $\sigma = 0$ is finished.

For the case $\sigma = 1$ we shall apply the lemma for $\sigma = 1$. The method used here is similar to the method used in the previous case.

Again replacing $B_{2m+2k+4}$ in (22) by formula (4) we get

$$\int_0^1 \frac{B_{2m+2}(x)}{(2m+2)!} \sum_{k=0}^{n} \frac{(2n+2k+3)!B_{2k+2}(x)}{(2k+1)!(2k+2)!(2n-2k+1)!}\, dx = \frac{(2n+2)!}{(4n+5)!2}\, \delta_{mn},\quad 0 \leq m \leq n. \tag{31}$$

Setting

$$\sum_{k=0}^{n} \frac{(2n+2k+3)!B_{2k+2}(x)}{(2k+1)!(2n-2k+1)!(2k+2)!} = P^*_{2n+2}(x), \tag{32}$$

then $P^*_{2n+2}(x)$ is a polynomial of precise degree $2n+2$ and we have

$$\int_0^1 B_{2m+2}(x)\, P^*_{2n+2}(x)dx = \frac{\{(2n+2)!\}^2\delta_{mn}}{(4n+5)!2},\ 0 \leq m \leq n. \tag{33}$$

Completely similar to the corresponding step in the proof of the theorem in case $\sigma = 0$ we also obtain

$$\int_0^1 B_{2m+1}(x)\, P^*_{2n+2}(x)dx = 0,\quad 0 \leq m \leq n \tag{34}$$

and

$$\int_0^1 B_0(x)\, P^*_{2n+2}(x)dx = 0 \tag{35}$$

Now, from (33), (34) and (35) and reasoning the same as before we can conclude that there exists a sequence of scalars $\{\beta_n\}$ with

$$P^*_{2n+2}(x) = \beta_n P_{2n+2}(2x-1),\quad n \geq 0. \tag{36}$$

We shall find the coefficients β_n. From (2) we have

$$\int_0^1 P_{2n+2}^2(2x-1)dx = \frac{1}{4n+5},$$

hence

$$J = \int_0^1 P_{2n+2}^{*2}(x)dx = \frac{\beta_n^2}{4n+5}. \qquad (37)$$

On the other hand, the following equalities are easy to be verified

$$J = \int_0^1 P_{2n+2}^{*2}(x)dx =$$

$$\sum_{k=0}^{n} \frac{(2n+2k+3)!}{(2k+1)!(2n-2k+1)!(2k+2)!} \cdot$$

$$\sum_{h=0}^{n} \frac{(2n+2h+3)!}{(2h+1)!(2n-2h+2)!(2h+2)!} \cdot$$

$$\int_0^1 B_{2k+2}(x)\, B_{2h+2}(x)dx$$

$$= \sum_{k=0}^{n} \frac{(2n+2k+3)!}{(2k+1)!(2n-2k+1)!} \cdot$$

$$\sum_{h=0}^{n} \frac{-(2n+2h+3)!B_{2k+2h+4}}{(2h+1)!(2n-2h+1)!(2k+2h+4)!}$$

Applying formula (22) to the inner finite sum, we obtain

$$J = \sum_{k=0}^{n} \frac{(2n+2k+3)!}{(2k+1)!(2n-2k+1)!} \cdot \frac{(2n+2)!\delta_{kn}}{(4n+5)!2}$$

$$= \frac{(4n+3)!(2n+2)!}{(2n+1)!(4n+5)!2} = \frac{1}{4(4n+5)}. \qquad (38)$$

From (36) we have $\beta_n > 0$, hence comparing (37) and (38) we have

$$\beta_n = 1/2 \,. \qquad (39)$$

Consequently, from (36) we get

$$P_{2n+2}(2x-1) = 2P_{2n+2}^{*}(x) =$$

$$2 \cdot \sum_{k=0}^{n} \frac{(2n+2k+3)!B_{2k+2}(x)}{(2k+1)!(2k+2)!(2n-2k+1)!}, \qquad (40)$$

and the proof of the theorem is finished.

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