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USE OF LAURICELLA'S HYPERGEOMETRIC FUNCTION $F_D^{(3)}$ IN THE SOLUTION OF TRIPLE INTEGRAL EQUATIONS

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ABSTRACT

In this paper the authors have presented the formal solutions of triple integral equations associated with Hankel kernels. It has been shown by the application of a known integral involving products of Bessel function and the known solutions of Abel's integral equation that the problem reduces to that of solving Fredholm integral equation to the first kind which can be easily solved by numerical methods. The results given earlier by Cooke and Tranter follow, as special cases.

RESUMEN

En este trabajo se considera la solución formal de ecuaciones integrales triples asociados con los kernels Hankel. Una integral conocida que involucran los productos de funciones de Bessel y la ecuación integral de Abel han sido utilizado para reducir el problema a una ecuación de tipo Fredholm de primera clase la cual puede ser resuelta fácilmente por métodos numéricos. Los resultados de Tranter y Cooke resultan como casos particulares.

1. INTRODUCTION

Formal solutions of certain triple integral equations with Hankel kernels has been the subject of interest during the last two decades and several authors have contributed a number of papers notably by Tranter (1960), Cooke (1963, 1963/64, 1965), Williams (1963), Lowndes (1969) and Noble (1958, 1963).

The method followed in obtaining the formal solutions of the integral equations (2.1), (2.2) and (2.3) given in Section 2 is essentially due to Noble (1958).

2. TRIPLE INTEGRAL EQUATIONS AND THEIR SOLUTIONS

The following triple integral equations will be solved here.

$$\int_0^\infty u^p H(u) J_\alpha(ux)du = 0 \quad (0 < x < a) \qquad \ldots(2.1)$$

$$\int_0^\infty u^q H(u) J_\beta(ux)du = f(x), \quad (a < x < b) \qquad \ldots(2.2)$$

$$\int_0^\infty u^p H(u) J_\alpha(ux)du = 0, \quad (b < x < \infty) \qquad \ldots(2.3)$$

where p and q are given constants and H(u) is an unknown function of u to be determined.

Two forms of the formal solutions will be obtained by employing different methods. For the sake of brevity, we will merely outline the proofs.

The first form of the formal solution of the triple integral equations obtained is

$$H(u) = u^{1-p} \int_a^b r\phi(r) J_\alpha(ur)dr \qquad \ldots(2.4)$$

where

$$\phi(r) = \frac{2\sin\pi\left(\frac{\beta+q-\alpha-p}{2}\right)}{\pi} r^{\alpha-1} \frac{d}{dr} \int_r^b \frac{s\, G(s)ds}{(s^2-r^2)^{\frac{\alpha+p-\beta-q}{2}}} \qquad \ldots(2.5)$$

and

$$s^{\alpha+\beta+q-p+1} G(s) = \frac{\Gamma\left(\frac{\beta+p-\alpha-q}{2}\right) \Gamma\left(\frac{\alpha+p-\beta-q}{2}\right) \sin\pi\left(\frac{\beta+p-\alpha-q}{2}\right)}{\pi 2^{q-p+1}}$$

$$\times \frac{d}{ds} \int_a^s \frac{x^{\beta+1} f(x)dx}{(s^2-x^2)^{\frac{\beta+p-\alpha-q}{2}}}$$

$$+ \left(\frac{2}{\pi}\right)^2 \frac{\sin\pi\left(\frac{\beta+q-\alpha-p}{2}\right)\sin\pi\left(\frac{\beta+p-\alpha-q}{2}\right)s}{(s^2-a^2)^{\frac{\beta+p-\alpha-q}{2}}}$$

$$\times \int_a^b \frac{tG(t)K(s,t)dt}{(t^2-a^2)^{\frac{\alpha+p-\beta-q}{2}}} \qquad \ldots(2.6)$$

where

$$K(s,t) = \frac{a^{\alpha+\beta+p-q+2}\,\Gamma\left(\frac{\alpha+\beta+q-p+2}{2}\right)\Gamma(p-q+1)}{2s^2t^2\,\Gamma\left(\frac{\alpha+\beta+p-q+4}{2}\right)}$$

$$\times F_1\left(\frac{\alpha+\beta+q-p+2}{2}; 1,1; \frac{\alpha+\beta+p-q+4}{2}; \frac{a^2}{s^2}, \frac{a^2}{t^2}\right) \ldots(2.7)$$

with $R(\alpha + \beta + q - p + 2) > 0$, $R(p - q + 1) > 0$,

$\left|\frac{a^2}{s^2}\right| < 1$, $\left|\frac{a^2}{t^2}\right| < 1$.

(2.7) is a Fredholm integral equation of the first kind, which can be solved numerically. Knowing $G(s)$, we can obtain $\emptyset(r)$ from (2.5) and $H(u)$ can then be calculated from (2.4).

The second form of the formal solution that will be developed here is

$$H(u) = u^{1-q} \int_0^a rf_1(r)\, J_\beta(ur)dr + \int_a^b rf(r)J_\beta(ur)dr$$

$$+ \int_b^\infty rf_2(r)J_\beta(ur)dr \qquad \ldots(2.8)$$

where

$$f_1(r) = \frac{2}{\pi}\sin\pi\left(\frac{\beta+q-\alpha-p}{2}\right)r^\beta(a^2-r^2)^{\frac{\alpha+p-\beta-q}{2}}$$

$$\times \left[-\int_a^b \frac{t^{-\beta+1}(t^2-a^2)^{\frac{\beta+q-\alpha-p}{2}} f(t)dt}{t^2-r^2}\right.$$

$$\left.\int_b^\infty \frac{t^{-\beta+1}(t^2-a^2)^{\frac{\beta+q-\alpha-p}{2}} f_2(t)dt}{t^2-r^2}\right], \ldots(2.9)$$

$(0 < r < a)$

and

$$f_2(r) = r^{-\beta}(r^2-b^2)^{\frac{\alpha+p-\beta-q}{2}} \left\{\frac{2}{\pi}\sin\pi\left(\frac{\beta+q-\alpha-p}{2}\right)\right\}^2$$

$$\times \left[\int_a^b f(y)K_2(y,r)dy - \int_a^b \frac{t^{\beta+1}(b^2-t^2)^{\frac{\beta+q-\alpha-p}{2}} f(t)dt}{r^2-t^2}\right.$$

$$\left.+ \int_0^\infty f(y)K_2(y,r)dy\right], \quad (b < r < \infty) \qquad \ldots(2.10)$$

where

$K_2(y,r)$

$$= \frac{y^{-\beta-1}(y^2-b^2)^{\frac{\beta+q-\alpha-p}{2}}\, a^{\alpha+\beta+p-q+2}\,\Gamma(\beta+1)\,\Gamma\left(\frac{\alpha+p-\beta-q+2}{2}\right)}{2r^2\, b^{\alpha+p-\beta-q}\,\Gamma\left(\frac{\alpha+\beta+p-q+4}{2}\right)}$$

$$\times F_D^{(3)}\left(\beta+1,1,1,\frac{\alpha+p-\beta-q}{2};\frac{\alpha+\beta+p-q+4}{2};\frac{a^2}{y^2},\frac{a^2}{r^2},\frac{a^2}{b^2}\right),$$

$\ldots(2.11)$

with $R(\beta+1) > 0$, $R(\alpha + p - \beta - q + 2) > 0$,

$$\left| \frac{a^2}{y^2} \right| < 1, \quad \left| \frac{a^2}{r^2} \right| < 1, \quad \left| \frac{a^2}{b^2} \right| < 1.$$

PROOF OF THE FIRST FORM

In view of (2.1), (2.3) and the Hankel's inversion formula it is observed that the unknown function $H(u)$ can be represented as (2.4). We now substitute the value of $H(u)$ from (2.4) in (2.2), invert the order of integration, apply the formula Erdélyi, A. et al (1954, p. 48) and Euler's integral hypergeometric function. Further we suppose that

$$G(s) = \int_s^b \frac{r^{-\alpha+1}\, \phi(r)dr}{(r^2-s^2)^{\frac{\beta+q-\alpha-p+2}{2}}},$$

then on applying the variants of Abel's integral equation Sneddon (1966, p. 41) the desired results (2.4) and (2.6) are obtained.

Let

$$\int_0^\infty u^q H(u) J_\beta(ux)du = f_1(x), \quad (0 < x < a)$$

$$= f_2(x), \quad (b < x < \infty)$$

then $H(u)$ can be represented by the equation (2.8). From (2.1), (2.3) and (2.8) it is found that

$$I_1(x) + I_0(x) + I_2(x) = 0, \quad (0<x<a;\ b<x<\infty)$$

where

$$I_j = \int r f_j(r) R(x,r)dr, \quad j = 0,1,2;$$

$f_0(r) = f(r)$, the limits being from a to b, 0 to a and b to ∞ respectively, and

$$R(x,r) = \int_0^\infty u^{1+p-\nu} J_\beta(ur)\, J_\alpha(ux)du$$

Now I_1 can be treated in the same way as in the proof of the first form. Then on applying the results Watson (1940, pp. 373, 401) and after evaluating the definite integrals involved in the analysis with the help of the formula :

$$\int_0^a y^c (a^2-y^2)^d \prod_{i=1}^{n} (s_i^2-y^2)dy$$

$$= \frac{a^{2d+c+1}}{2} \prod_{i=1}^{n} s_i^{-2\alpha_i} \frac{\Gamma(\frac{c+1}{2})\, \Gamma(d+1)}{\Gamma(\frac{2d+c+3}{2})}$$

$$\times F_D^{(n)} \left(\frac{c+1}{2}, \alpha_1,\ldots,\alpha_n; \frac{2d+c+3}{2}; \frac{a^2}{a_1^2},\ldots,\frac{a^2}{a_n^2}\right)$$

where $R(c+1) > 0$, $R(d+1) > 0$, $\left| \frac{a^2}{s_i^2} \right| <$

for $i = 1,\ldots,n$,

we arrive at the result.

For a detailed discussion of Lauricella function

$$F_D^{(n)}(a,b_1,\ldots,b_n;\ c;\ x_1,\ldots,x_n)$$

the readers are referred to the monograph by Exton (1976, p. 41)

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REFERENCES

1) APPELL, P. and KAMPE DE FERIET, J. : "*Functions Hypergeometriques et Hyperspheriques*", Gauthier-Villars, Paris, 1926.

2) COOKE, J.C. : "*Triple integral equations*". Quart. J. Mech. Appl. Math. 16, 1963, pp. 193-203.

3) COOKE, J.C. : "*Some further triple integral equations solutions*". Proc. Edinburgh Math. Soc. (2), 13 (1963-64), pp. 305-316.

4) COOKE, J.C. : "*The solution of triple integral equations in operational form*". Quart. J. Mech. Appl. Math. 18, 1965, 57-62.

5) ERDELYI, A. ET AL : "*Higher transcendental functions*", Vol. 1, McGraw-Hill, New York, 1953.

6) ERDELYI, A. ET AL. : "*Tables of integral transforms*", Vol. 2, McGraw-Hill, New York, 1954.

7) EXTON, H. : "*Multiple hypergeometric functions and applications*". Ellis Horwood Ltd.Chichester, 1976.

8) LOWNDES, J.S.: "*Dual series and triple integral equations*". Proc. Edinburgh Math. Soc. 16, 1969, pp. 273-280.

9) NOBLE, B. : "*Certain dual integral equations*".J. Math. Phys. 37, 1958, pp. 128-136.

10) NOBLE, B. : "*The solution of Bessel function dual integral equations by a multiplying factor method*". Prod. Cambridge Philos. Soc. 59, 1963, p. 351.

11) SNEDDON, I.N. : "*Mixed boundary value problems in potential theory*". North-Holland, 1966.

12) TRANTER, C.J. : "*Some triple integral equations*". Proc. Glasgow Math. Assoc. 4, 1960, p. 200.

13) WATSON, G.N.: "*Theory of Bessel functions*".Cambridge, 1940.

14) WILLIAMS, W.E. : Quart. J. Mech. Appl.Math. 16, 1963, p. 204.

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