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GLOBALLY FRAMED PARA F-STRUCTURE MANIFOLDS

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ABSTRACT

In this note, the definition of a globally framed para f-structure manifold is given. It is shown that an almost product structure, or a para-contact structure, can be defined, on a globally framed para f-structure manifold, depending on the number of the vector fields in the characteristic equation.

RESUMEN

En este trabajo se da la definición de una variedad con estructura para f enmarcada. Se demuestra que una estructura casi producto o una estructura para contacto se puede definir en esta variedad dependiendo del número de campos vectoriales en la ecuación característica.

1. INTRODUCCION

A C^∞, n-dimensional manifold M, on which there exists a C^∞, tensor field $F\neq 0$, of type (1,1), such that

$$F^2 = -I \qquad (1.1)$$

where I is the identity tensor on M, is called an almost complex manifold with an almost complex structure F. Yano [6].

If equation (1.1) is replaced by

$$\text{(a) } F^2 + I = u \otimes U, \text{ (b) } FU = 0, \text{ (c) rank } F = n-1 \qquad (1.2)$$

where U and u are C^∞ vector field and 1-form on M respectively, then M is called an almost contact manifold, with almost contact structure, Boothby and Wong [1].

If equation (1.1), is replaced by

$$\text{(a) } F^3 + F = 0, \text{ (b) rank } F = r, \quad 1 \le r \le n \qquad (1.3)$$

then F is called an F-structure, and M is called an F-structure manifold. On an F-structure manifold, let

$$\text{(a) } \ell = -F^2, \text{ (b) } m = F^2 + I \qquad (1.4)$$

then ℓ and m are complementry projection operators acting on the tangent space at each point of M. They define two differentiable distributions, P and Q on M, such that dim P= r, and dim Q= n-r. Yano [7].

If on an F-structure manifold M, there are n-r, C^∞ vector fields U_a, spanning Q at each point of M, and there are C^∞, n-r, 1-forms u^a, satisfying

$$F^2 + I = u^a \otimes U_a \text{ (summation), (b) } FU_b = 0, \qquad (1.5)$$

a,b= 1,...,n-r, then M is said to be a globally framed F-structure manifold. If a Riemannian metric h is defined on a globally framed F-structure manifold by $u^a = h(U_a,.)$, then M is called a globally framed metric F-structure manifold. Goldberg and Yano [3].

In another paper, Goldberg and Yano [2], showed that an almost complex structure can be defined on a globally framed F-structure manifold M, if the dimension of M is even. If the dimension of M is odd, then an almost contact structure can be defined.

If equation (1.1) is replaced by

$$F^2 = I \qquad (1.6)$$

then M is said to be an almost product manifold, with almost product structure F. Yano [6].

If equation (1.6) is replaced by

$$\text{(a) } F^2 = I - u \otimes U, \text{ (b) } FU = 0, \text{ (c) rank } F = n-1 \qquad (1.7)$$

then M is called a para-contact manifold, with para-contact structure, Sato [5].

Let M be a C^{∞}, n-dimensional manifold, let $f \neq 0$, be a C^{∞}, tensor field of type (1,1) on M, satisfying

$$\text{(a) } f^2 = d^2 I + c\, u^a \otimes U_a \text{ (summation)},$$

$$\text{(b) } fU_a = P^b_a U_b, \qquad (1.8)$$

where d^2, c are constants, a,b= 1,2,...,n-r, ($1 \le r \le n$), U_a are linearly independent C^{∞} vector fields, u^a are C^{∞} 1-forms, and P^b_a are scalers, then M is called a generalized structure manifold, with a generalized F-structure, Mishra [4].

2. GLOBALLY FRAMED PARA F-STRUCTURE MANIFOLD

In equation (1.8), let $d^2 = 1$, $c = -1$, $P^b_a = 0$, and rank f= r, then

$$f^2 = I - u^a \otimes U_a \text{ , (b) } fU_a = 0 \text{ , } a = 1,\ldots,n-r. \qquad (2.1)$$

In this case M is called a globally framed para f-structure manifold, with a para f-structure.

Note 1 :- When a = 1, then M is a para-contact manifold.

Theorem 1.

On a globally framed para f-structure manifold, we have

(i) $u^a(U_b) = \delta^a_b$, δ is the Kronecker delta.

(ii) $u^a \circ f = 0$.

Proof:-

(i) From equation (1.9) (a), we have

$$f^2(U_b) = U_b - u^a(U_b)U_a$$

But $fU_b = 0$, $\Rightarrow U_b = u^a(U_b)U_a$

$$\therefore u^a(U_b) = \delta^a_b$$

(ii) Premultiply and postmultiply (1.9) (a), by f, and use (1.9) (b), we have

$$f^3 = f - u^a \circ f \otimes U_a \qquad (1)$$

and $$f^3 = f - u^a \otimes fU_a \qquad (2)$$

from (1) and (2) we get $u^a \circ f = 0$.//

Theorem 2.

The operators given by

$$\ell = f^2 \text{ , } m = -f^2 + I = u^a \otimes U_a$$

applied on M_p, $p \in M$, are complementry projection operators, and we are going to have two complementry tangent sub-bundles L and N or with dim L=r, and dim N= n-r.

Proof:-

$$\ell + m = I \qquad (1)$$

Multiply (1.9) (a) by f, we have

$$f^3 = f \qquad (2)$$

$\ell^2 = f^4 = f^2 = \ell$, from (2).

Premultiply (1) by ℓ, we have

$\ell^2 + \ell m = \ell \Rightarrow m = 0$. Similarly $m\ell = 0$.

Multiply (1) by m, we have

$$m\ell + m^2 = m \Rightarrow m^2 = m.$$

Since rank f = r, everywhere on M, and $m = u^a \otimes U_a$, (a= 1,...,n-r), then we are going to have two complementry tangent sub-bundles L, and N corresponding to ℓ and m respectively, such that dim L = r, and dim N= n-r. N is spanned by the vector fields $U_1,\ldots,U_{n-r}$.//

3. ALMOST PRODUCT AND PARA-CONTACT STRUCTURES ON M

Theorem 3.

Let M be a para f-structure manifold, suppose that n-r is an even integer, i.e., the number of the vector fields U_a is even, then an almost product structure is defined on M.

Proof:-

Put

$$J = f + u^{2i} \otimes U_{2i-1} + u^{2i-1} \otimes U_{2i} \quad \text{(summation)},$$

$i = 1, \ldots, \frac{n-r}{2}$

$J(X) = f(X) + u^{2i}(X)U_{2i-1} + u^{2i-1}(X)U_{2i}$; X is a vector field.

$$J^2(X) = f^2(X) + u^{2i}[f(X)]U_{2i-1} + u^{2i-1}[f(X)]U_{2i}$$

$$+ u^{2i}(X)(fU_{2i-1}) + u^{2i}(X)u^{2i}(U_{2i-1})U_{2i-1}$$

$$+ u^{2i}(X)u^{2i-1}(U_{2i-1})U_{2i} + u^{2i-1}(X)(fU_{2i})$$

$$+ u^{2i-1}(X)u^{2i}(U_{2i})U_{2i-1} + u^{2i-1}(X)u^{2i-1}(U_{2i})U_{2i}$$

$u^a \circ f = 0 = u^{2i}[f(x)] = 0$ and $u^{2i-1}[f(X)] = 0$

$fU_a = 0 \Rightarrow fU_{2i-1} = 0$ and $fU_{2i} = 0$

$$u^a(U_b) = \delta^a_b \Rightarrow u^{2i}(U_{2i-1}) = 0, \; u^{2i-1}(U_{2i}) = 0,$$

$$u^{2i}(U_{2i}) = 1, \; u^{2i-1}(U_{2i-1}) = 1$$

$$\therefore J^2(X) = f^2(X) + u^{2i}(X)U_{2i} + u^{2i-1}(X)U_{2i-1}$$

$$= f^2(X) + u^a(X)U_a, \quad a = 1, \ldots, n-r$$

From (1.9) (a) we have

$$f^2(X) + u^a(X)U_a = X$$

$$\therefore J^2(X) = X, \text{ or } J^2 = I.//$$

Theorem 4.

Let M be a globally framed para f-structure monifold, suppose that n-r is an odd integer, then a para-contact structure is defined on M.

Proof:-

Put

$$J = f + u^{2i} \otimes U_{2i-1} + u^{2i-1} \otimes U_{2i}, \quad i = 1, \ldots, \frac{n-r-1}{2}$$

Similar to the prove of theorem (3) we have for any vector field X in M

$$J^2(X) = f^2(X) + u^a(X)U_a, \quad a = 1, 2, \ldots, n-r-1$$

$$\therefore J^2(X) = f^2(X) + u^a(X)U_a + u^{n-r}(X)U_{n-r} - u^{n-r}(X)_{n-r}$$

$$= f^2(X) + u^b(X)U_b - u(X)U$$

where $b = 1, \ldots, n-r$, $u = u^{n-r}$, and $U = U_{n-r}$

$$\therefore J^2(X) = X - u(X)U = J^2 = I - u \otimes U$$

J is a para-contact structure on M.//

4. METRIC ON GLOBALLY FRAMED PARA F-STRUCTURE MANIFOLD

Let us introduce a metric tensor g on a globally framed para f-structure manifold M, such that

$g(fX,Y) + g(X,fY) = 0$, X, Y are any two vector fields, then M is called a globally framed metric para f-structure manifold.

In equation (4.1) replace Y by fY, we get

$$g(fX,fY) = -g(X,f^2Y) \tag{4.2}$$

But $f^2 = I - u^a \otimes U_a$.

Substitute for f^2 in equation (4.2), we have

$$g(fX,fY) = g(X,u^a(Y)U_a - Y) = u^a(Y)g(X,U_a) - g(X,Y)$$

Therefore,

$$g(fX,fY) = u^a(Y)u^a(X) - g(X,Y) \tag{4.3}$$

where $u^a(X) = g(X,U_a)$.

Theorem 5.

On a globally framed metric f-structure manifold, the complementry tangent sub-bundles L and N are orthogonal with respect to the metric g.

Proof:-

L and N correspond to the complementry projection operators ℓ and m, therefore,

$$f(\ell X,mY) = g(f^2X,u^a(Y)U_a) = u^a(Y)g(f^2X,U_a).$$

In equation (1.3), replace X by fX, we get

$$g(f^2X,Y) + g(fX,fY) = 0$$

$$\therefore\ g(\ell X,mY) = -u^a(Y)g(fX,fU_a) = 0,$$

since $fU_a = 0$, $a = 1,\dots,n-r$.//

5. EXAMPLES

Example 1.

Consider the 7-dimensional Euclidean space V_7, on which a tensor of type (1,1) is defined as

$$f^2 = \begin{bmatrix} 0 & & & & & & \\ & 0 & & & & & \\ & & 0 & & & 0 & \\ & & & 0 & & & \\ & & 0 & & -1 & & \\ & & & & & -1 & \\ & & & & & & 1 \end{bmatrix} = f^2 = \begin{bmatrix} 0 & & & & & & \\ & 0 & & & & 0 & \\ & & 0 & & & & \\ & & & 0 & & & \\ & 0 & & & 1 & & \\ & & & & & 1 & \\ & & & & & & 1 \end{bmatrix}$$

$$f^2 = I_7 - \begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ & & & & 0 & & \\ & & & & & 0 & \\ & & & & & & 0 \end{bmatrix} =$$

$$= I_7 - [1000000] \otimes [1000000]^t$$

$$- [0100000] \otimes [0100000]^t - [0010000] \otimes [0010000]^t$$

$$- [0001000] \otimes [0001000]^t = I_7 - u^a \otimes U_a$$

t means "transpose".

Further, $fU_b = 0$, $a,b = 1,2,3,4$, and rank $f = 3$. Hence, V_7 is a globally framed para f-structure manifold.

Consider, the new tensor F on V_7 defined by

$$F = f + u^{2i} \otimes U_{2i-1} + u^{2i-1} \otimes U_{2i},\quad i = 1,2$$

$$F = f + [0100000] \otimes [1000000]^t +$$

$$+ [0001000] \otimes [0010000]^t$$

$$+ [1000000] \otimes [0100000]^t +$$

$$+ [0010000] \otimes [0001000]^t .$$

Let $X = [x_1,\dots,x_7]^t$ be any vector field on V_7,

$$FX = [x_2,x_1,x_4,x_3,-x_5,-x_5,-x_6,x_7]^t ,$$

$$F^2X = X$$

i.e. $F^2 = I$, i.e.: V_7 admits an almost products structure.

Example 2.

Consider the Euclidean space V_5 on which

$$f = \begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & -1 & \\ & & & & 1 \end{bmatrix} \Rightarrow f^2 = \begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$$

$$f^2 = I_5 - [10000] \otimes [10000]^t -$$

$$- [01000] \otimes [01000]^t$$

$$- [00100] \otimes [00100]^t = I_5 - u^a \otimes U_a$$

$fU_b = 0$, $a,b = 1,2,3$, and rank $f = 2$.

V_5 is a globally framed para f-structure monifold.

Consider

$$F = f + u^{2i} \otimes U_{2i-1} + u^{2i-1} \otimes U_{2i} ,\quad i = 1$$

$$F = f + [01000] \otimes [10000]^t + [10000] \otimes [01000]^t .$$

Let $X = [x_1,x_2,x_3,x_4,x_5]^t$ be any vector field.

$$FX = [000, -x_4, x_5]^t + [x_2, 0000]^t + [0,x_1,000]^t$$

$$F^2X = [000,\ x_4,\ x_5]^t + [0,x_2,000]^t + [x_1,0000]^t$$

$$= [x_1,x_2,0,x_4,x_5]^t = [x_1,x_2,x_3,x_4,x_5]$$

$$- [000\ x_3\ \ 00]$$

$$F^2X = X - [000100]\ ([x_1,x_2,x_3,x_4,x_5]^t)\ [000100]^t$$

$$F^2X = X - u^3(X)U_3 = F^2 = I_5 - u^3 \otimes U_3$$

This means that V_5 admits a para-contact f-structure.

Note 2:-

Examples (1) and (2) can be easily generalized to any Euclidean space V_n on which

$$f = \begin{bmatrix} 0 & & 0 \\ & -I_p & \\ 0 & & I_q \end{bmatrix}$$

Example 3.

Consider the Euclidean space V_6 on which

$$f = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$f^2 = \begin{bmatrix} 0 & 0 \\ 0 & I_4 \end{bmatrix} = I_6 - \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= I_6 - [100000] \otimes [100000]^t -$$

$$- [010000] \otimes [010000]^t = I_6 - u^a \otimes U_a$$

rank $f = 4$, $fU_j = 0$, $a,j = 1,2$.

Hence V_6 is a globally framed para f-structure manifold.

Define

$$F = f + u^{2i} \otimes U_{2i-1} + u^{2i-1} \otimes U_{2i}\ ,\quad i = 1,2$$

$$= f + [010000] \otimes [100000]^t +$$

$$+ [100000] \otimes [010000]^t$$

Let $X = [x_1,x_2,x_3,x_4,x_5,x_6]^t$

$$FX = [0,0,-x_5,x_6,-x_3,x_4] + [x_2,00000] +$$

$$+ [0,x_1,0000] = [x_2,x_1,-x_5,x_6,-x_3,x_4]$$

$$F^2X = [0,0,x_3,x_4,x_5,x_6] + [0,x_2,0000] + [x_1,00000]$$

$$= X \Rightarrow F^2 = I\ .$$

Hence, V_6 admits on almost product structure.

Example 4.

Let M be any globally framed f-structure manifold given by the characteristic equation

$$F^2 + I = u^a \otimes U_a\ ,\qquad a = 1,2,\dots,n-r\ .$$

Put $f = F^2$ then

$$f + I = u^a \otimes U_a\ .$$

Post multiply by f we have

$$f^2 + f = u^a \otimes fU_a = u^a \otimes F^2 U_a = 0$$

$$f^2 + f = 0 = f^2 + u^a \otimes U_a - I = 0$$

$$f^2 = I - u^a \otimes U_a$$

which is a globally framed para f-structure.

The same result we get if we put $f = -F^2$.

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