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A BASIC ANALOGUE OF H-FUNCTION OF TWO VARIABLES

ABSTRACT

In course of an attempt to unify and extend the results of basic hypergeometric functions, the authors define a basic analogue of H-function of two variables and investigate some of its main properties. The basic analogue of H-function of two variables is extended to several variables.

RESUMEN

En un intento para unificar y extender los resultados de funciones hipergeométricas básicas, los autores definen una analogía básica de la función H de dos variables e investigan algunas de sus principales propiedades. La analogía básica de la función H de dos variables es extendida a varias variabes.

1. INTRODUCTION

The G- and H-functions have been studied extensively by several authors [2,3,7]. Munot and Kalla [4] have extended the H-function in the domain of two variables, whereas Saxena [5] and Srivastava [7] have treated the case of several variables.

The great success of the theory of hypergeometric functions in one and various variables has stimulated the development of a corresponding basic analogue of these functions. Let q be a parameter which in general shall be restricted to the domain $|q| < 1$, and

$$(q)_{q,n} = (1-q)(1-q^2)\ldots(1-q^n),\ n=1,2,\ldots$$

$$(q)_{q,0} = 1 \tag{1.1}$$

$$(a_i)_{q,n} = (1-q^{a_i})(1-q^{a_i+1})\ldots(1-q^{a_i+n-1}),\ n=1,2,\ldots$$

$$(a_i)_{q,0} = 1. \tag{1.2}$$

Then

$$ {}_r\Phi_s\left[\begin{matrix}\alpha_1,\ldots \alpha_r\\ \rho_1,\ldots \rho_s\end{matrix};z\right] = $$

$$= \sum_{n=0}^{\infty} \frac{(\alpha_1)_{q,n}(\alpha_2)_{q,n}\ldots(\alpha_r)_{q,n}}{(q)_{q,n}(\rho_1)_{q,n}\ldots(\rho_s)_{q,n}}\, z^n \tag{1.3}$$

$$|z|<1$$

is a function of z and of r+s+1 parameters $\alpha_1,\ldots,\alpha_r;\ \rho_1,\ldots\rho_s;\ q$, which reduces to a generalized hypergeometric series ${}_rF_s$, if r=s+1 and $q\to 1$. ${}_r\Phi_s$ is called a basic hypergeometric series. It should be observed that the notations in [A. Erdélyi : Higher Transcendental Functions, Vol. I, McGraw-Hill, New York (1953), pp. 195] have not been explained adequately.

In this paper we introduce a basic analogue of H-function of two variables in the theory of generalized hypergeometric series, which is an extension of the basic H-function defined earlier by Saxena, Modi and Kalla [6]. Some of its main properties are established. The results are then extended to the case of several variables.

2. DEFINITION OF A BASIC H-FUNCTION OF TWO VARIABLES

A basic analogue of H-function of two variables [4] is defined in terms of a double Mellin-Barnes type integral as :

$$H^{A,(M_1:N_1),(M_2:N_2)}_{C,D,(P_1:Q_1),(P_2:Q_2)}\left[\begin{matrix} z_1 \\ z_2 \end{matrix} ; q \left| \begin{matrix} (c_j;\gamma_j,\gamma'_j) \\ (a_j,\alpha_j);(a'_j,\alpha'_j) \\ (d_j;\delta_j,\delta'_j) \\ (b_j,\beta_j);(b'_j,\beta'_j) \end{matrix} \right.\right]$$

$$= \frac{1}{(2\pi i)^2} \int_{q^*} \int_{q^*} x_1(s_1;q) x_2(t;q) x_3(s,t;q) \times$$

$$\times \frac{\pi^2 z_1^s z_2^t \, ds \, dt}{G(1-s)G(1-t)\sin\pi s \sin\pi t} \tag{2.1}$$

where $|q|<1$, $\log q=-w=-(w_1+iw_2)$ where w, w_1, w_2 are constants, w_1 and w_2 being real. Further

$$x_1(s;q) = \frac{\prod_{j=1}^{M_1} G(b_j-\beta_j s) \prod_{j=1}^{N_1} G(1-a_j+\alpha_j s)}{\prod_{j=M+1}^{Q_1} G(1-b_j+\beta_j s) \prod_{j=N_1+1}^{P_1} G(a_j-\alpha_j s)} ;$$

$$G(\alpha) = \{ \prod_{n=o}^{\infty} (1-q^{\alpha+n}) \}^{-1}.$$

$$x_2(s;q) = \frac{\prod_{j=1}^{M_2} G(b'_j-\beta'_j s) \prod_{j=1}^{N_2} G(1-a'_j+\alpha'_j s)}{\prod_{j=M_2+1}^{Q_2} G(1-b'_j+\beta'_j s) \prod_{j=N_2+1}^{P_2} G(a'_j-\alpha'_j s)}$$

$$x_3(s,t;q) = \frac{\prod_{j=1}^{A} G(1-c_j+\gamma_j s+\gamma'_j t)}{\prod_{j=A+1}^{C} G(c_j-\gamma_j s-\gamma'_j t) \prod_{j=1}^{D} G(1-d_j+\delta_j s+\delta'_j t)}$$

γ_j and $\gamma'_j (1\le j\le C)$; δ_j, δ'_j $(1\le j\le D)$; $\alpha_j (1\le j\le P_1)$, α'_j $(1\le j\le P_2)$, $\beta_j (1\le j\le Q_1)$, $\beta'_j(1\le j\le Q_2)$ are positive numbers, A, D, C, P_1, P_2, Q_1, Q_2, M_1, M_2, N_1 and N_2 are non-negative integers, satisfying the following inequalities $0\le A\le C$, $0\le M_i\le Q_i$, $0\le N_i \le P_i$, $D>0$; $\forall i\in\{1,2\}$. The contours C_1^* and C_2^* are lines parallel to $Rl(w_i s) = 0$ $(i = 1,2)$ with indentations, if necessary, in such a manner that all the poles of $G(b_j-\beta_j s)$ for $j \in \{1,\dots,M_1\}$ and $G(b'_j-\beta'_j t)$ for $j \in\{1,\dots,M_2\}$ lie to right and those of $G(1-c_j+\gamma_j s+\gamma'_j t)$ for $j \in\{1,\dots,A\}$; $G(1-a_j+\alpha_j s)$ for $j \in\{1,\dots,N_1\}$ and $G(1-a'_j+\alpha'_j t)$ for $j \in\{1,\dots,N_2\}$; lie to the left of the contours. An empty product is interpreted as unity. The poles of the integrand are assumed to be simple.

The integrals converge if

$Rl\left[s \log(z_1)-\log \sin\pi s\right]< 0$ and $Rl\left[t \log z_2 - \log \sin\pi t\right] < 0$ for large values of $|t|$ and $|s|$ on the contours i.e. $|\{\arg(z_i)-w_1^{-1} w_2 \log |z_i|\}|<\pi$ for $i=1,2$. When $A = C = D = 0$, (2.1) reduces to a product of two basica analogue of Fox's H-function due to Saxena, Modi and Kalla [6]. The result is

$$H^{0,(M_1,N_1),(M_2,N_2)}_{0,0,(P_1,Q_1),(P_2,Q_2)}\left[\begin{matrix} z_1 \\ z_2 \end{matrix} ;q \left| \begin{matrix} (a_j,\alpha_j);(a'_j,\alpha'_j) \\ (b_j,\beta_j);(b'_j,\beta'_j) \end{matrix} \right.\right]$$

$$= H^{M_1,N_1}_{P_1,Q_1}\left[z_1;q \left| \begin{matrix} (a_j,\alpha_j) \\ (b_j,\beta_j) \end{matrix} \right.\right] H^{M_2,N_2}_{P_2,Q_2}\left[z_2;q \left| \begin{matrix} (a'_j,\alpha'_j) \\ (b'_j,\beta'_j) \end{matrix} \right.\right] \tag{2.2}$$

If we make suitable changes in the parameters in (2.1), it can then give rise to the definitions of the basic analogue of several generalized special functions, such as G -function of two variables, Kampe de Feriet's function of two variables; Appell's function of two variables F_1, F_2, F_3 and F_4 and Whittaker functions of two variables, etc. For the sake of brevity they are not presented here.

From the definition (2.1), it is readily seen that

$$z_1^{\sigma_1} z_2^{\sigma_2} H^{A,(M_1:N_1),(M_2:N_2)}_{C,D,(P_1:Q_1),(P_2:Q_2)}\left[\begin{matrix} z_1 \\ z_2 \end{matrix} ; q\right]$$

$$= (-1)^{\sigma_1+\sigma_2} H^{A,(M_1+1:N_1),(M_2+1:N_2)}_{C,D,(P_1+1:Q_1+1),(P_2+1:Q_2+1)} \times \left[\begin{matrix} z_1 \\ z_2 \end{matrix} ; q \;\middle|\; \begin{matrix} (c_j;)\gamma_j,\gamma'_j) \\ (a_j,\alpha_j);(\alpha'_j,\alpha'_j) \\ (d_j;\ \delta_j,\ \delta'_j) \\ (b_j,\beta_j);(b_j,\beta'_j) \end{matrix} \right] d(q;x)$$

$$\left. \begin{matrix} (c_j+\gamma_j\sigma_1+\gamma'_j\sigma_2;\ \gamma_j,\ \gamma'_j) \\ (a_j+\alpha_j\sigma_1,\alpha_j),(1+\sigma_1,1);(a'_j+\alpha'_j\sigma_2,\alpha'_j),(1+\sigma_2,1) \\ (d_j+\delta_j\sigma_1+\delta'_j\sigma_2;\ \gamma_j,\ \gamma'_j) \\ (1,1),(b_j+\beta_j\sigma_1,\beta_j);(1,1),(b'_j+\beta'_j\ \sigma_2,\beta'_j) \end{matrix} \right] \qquad (2.3)$$

and

$$H^{o}\left[\begin{matrix} z_1^{-1} \\ z_2^{-1} \end{matrix} ;q \right] = H^{0,\ (N_1+1:M_1),(N_2+1,M_2)}_{D,C,(Q_1:P_1),(Q_2:P_2)} \times \left[\begin{matrix} z_1 \\ z_2 \end{matrix} ; q \;\middle|\; \begin{matrix} (1-d_j;\delta_j,\delta'_j) \\ (1-b_j,\beta_j);(1-b'_j,\beta_j) \\ (1-c_j;\gamma_j,\gamma'_j) \\ (1,1),(1-a_j,\alpha_j),(0,1);(1,1),(1-a'_j,\alpha'_j),(0,1) \end{matrix} \right] \qquad (2.4)$$

3. CERTAIN BASIC INTEGRALS INVOLVING H_q-FUNCTION OF TWO VARIABLES:

The following basic integrals will be established

$$\frac{G(1)}{1-q} \int_0^1 x^{\rho-1} E_q(q.x) H \left[\begin{matrix} z_1 x^{\sigma_1} \\ z_2 x^{\sigma_2} \end{matrix} ;q \;\middle|\right.$$

$$= H^{A+1,(M_1:N_1),(M_2:N_2)}_{C+1,D,(P_1:Q_1),(P_2:Q_2)} \left[\begin{matrix} z_1 \\ z_2 \end{matrix} ;q \;\middle|\; \begin{matrix} (1-\rho;\sigma_1,\sigma_2),(c_j;\gamma_j,\gamma'_j) \\ (a_j,\alpha_j);(a'_j,\alpha'_j) \\ (d_j;\delta_j,\ \delta'_j) \\ (b_j,\beta_j);(b'_j,\beta'_j) \end{matrix} \right] \qquad (3.1)$$

for $Rl(\rho)> 0$, $Rl(\rho_1)> 0$, $Rl(\rho_2)> 0$, $|\{\arg z_i - w_2 w_1^{-1} \times \times \log|z_i|\}| < \pi\ \forall i \in \{1,2\}$, $|q|< 1$.

$$\frac{G(1)}{1-q} \int_0^1 x^{\sigma-1}(1-qx)_{\rho-\sigma-1} H^{A,(M_1:N_1),(M_2:N_2)}_{C,D,(P_1:Q_1),(P_2:Q_2)} \left[\begin{matrix} z_1 x^{\mu_1} \\ z_2 x^{\mu_2} \end{matrix} ;q \;\middle|\; \begin{matrix} (c_j;\gamma_j,\ \gamma'_j) \\ (a_j,\alpha_j);(a_j,\alpha'_j) \\ (d_j;\delta'_j,\delta'_j) \\ (b_j,\beta_j);(b'_j,\beta'_j) \end{matrix} \right] d(q,x)$$

$$= G(\rho-\sigma) H^{A+1,(M_1,N_1),(M_2,N_2)}_{C+1,D+1,(P_1,Q_1),(P_2,Q_2)} \left[\begin{matrix} z_1 \\ z_2 \end{matrix} ;q \;\middle|\right.$$

$$\left.\begin{matrix}(1-\rho;\mu_1,\mu_2),(c_j;\gamma_j,\gamma_j')\\(a_j,\alpha_j);(a_j',\alpha_j')\\(d_j;\delta_j,\delta_j'),(1-\sigma;\mu_1,\mu_2)\\(b_j,\beta_j);(b_j',\beta_j')\end{matrix}\right] \quad (3.2)$$

for $Rl(\sigma) > 0$, $Rl(\mu_i) > 0$, $Rl(\rho-\sigma) > 0$, $|\{\arg z_i - w_2 w_1^{-1} x\log|z_i|\}| < \pi \forall i \in \{1,2\}, |q| < 1$.

$$\frac{1}{2\pi i}\int_c eq(x)x^{-\sigma} H^{A,(M_1,N_1),(M_2:N_2)}_{C,D,(P_1:Q_1),(P_2,Q_2)}\left[\begin{matrix}z_1x^{\rho_1}\\z_2x^{\rho_2}\end{matrix};q\;\middle|\;\begin{matrix}(c_j,\gamma_j,\gamma_j')\\(a_j,\alpha_j);(a_j',\alpha_j')\\(d_j;\delta_j,\delta_j')\\(b_j,\beta_j);(b_j',\beta_j')\end{matrix}\right]dx$$

$$= G(1)H^{A,(M_1:N_1),(M_2:N_2)}_{C+1,D,(P_1:Q_1),(P_2:Q_2)}\left[\begin{matrix}z_1\\z_2\end{matrix};q\;\middle|\;\begin{matrix}(c_j;\gamma_j,\gamma_j'),(\sigma;\rho_1,\rho_2)\\(a_j,\alpha_j);(a_j',\ j')\\(d_j;\delta_j,\delta_j')\\(b_j,\beta_j);(b_j',\beta_j')\end{matrix}\right] \quad (3.3)$$

where the path of integration C encircles the null-point and also in the usual manner, can be deformed into a loop parallel to the imaginary axis.

The proof of these integrals can be developed on similar lines as given by Saxena, Modi and Kalla [6], on employing the integrals due to Hahn [1].

4. A BASIC ANALOGUE OF H-FUNCTION OF SEVERAL VARIABLES:

A basic analogue of H-function of several complex variables [5] can be defined analogously. The definition is as follows :

$$H^{A,(M_n:M_n)}_{C,D,(P_n:Q_n)}\left[(z),q\;\middle|\;\begin{matrix}(c_j,(\gamma_j^{(n)}));(d_j,(\delta_j^{(n)}))\\((a_j^{(n)},\alpha_j^{(n)}));((b_j^{(r)},\beta_j^{(r)}))\end{matrix}\right]$$

$$= \frac{1}{(2\pi i)^n}\int_{C_1^*}\ldots(n)\ldots\int_{C_r^*} X(s_1,s_2,\ldots,s_n;q)$$

$$\prod_{i=1}^{n}\left\{X_i(s_i)\frac{ds_i \pi z_i^{s_i}}{G(1-s_i)\sin(\pi s_i)}\right\}$$

where

$$X(s,\ldots,s_n;q) =$$

$$\frac{\prod_{j=1}^{A} G(1-c_j+\sum_{i=1}^{n}\gamma_j^{(i)}s_i)}{\prod_{j=A+1}^{C} G(c_j-\sum_{i=1}^{n}\gamma_j^{(i)}s_i)\prod_{j=1}^{D} G(1-d_j+\sum_{i=1}^{n}\delta_j^{(i)}s_i)}$$

$$X_i(s_i) =$$

$$\frac{\prod_{j=1}^{M_i} G(b_j^{(i)}-\beta_j^{(i)}s_i)\prod_{j=1}^{N_i} G(1-a_j^{(i)}+\alpha_j^{(i)}s_i)}{\prod_{j=M_i+1}^{Q_i} G(1-b_j^{(i)}+\beta_j^{(i)}s_i)\prod_{j=N_i+1}^{P_i} G(a_j^{(i)}-\alpha_j^{(i)}s_i)}, \tag{4.i}$$

and $\gamma_j^{(i)}, 1\le j\le C, \delta_j^{(i)}, 1\le j\le D, \alpha_j^{(i)}, 1\le j\le P_i, 1\le j\le Q_i$, $i\in\{1,\dots,n\}$ are positive numbers. C_j, $1\le j\le C$; d_j, $1\le j\le D$; $a_j^{(i)}, 1\le j\le P_i$ and $b_j^{(i)}, 1\le j\le Q_i$ are complex numbers. A, C, D, P_i, Q_i, M_i and N_i are non-negative integers, satisfying the following inequalities $0\le A\le C$, $0\le M_i\le Q_i$, $0\le N_i\le P_i$, $D\ge 0$ $\forall$ $i\in\{1,\dots,n\}$, and $|q|<1$.

The contours C_i^*'s are lines parallel to $Re(w_i s) = 0, (i=1,\dots,n)$ with indentations, if necessary, in such a manner that all the poles of $G(b_j^{(i)} - \beta_j^{(i)} s_i)$ for $j\in\{1,\dots,M_i\}$ and $i\in\{1,\dots,n\}$ lie to right and those of $G(1-c_j + \sum_{i=1}^{n}\gamma_j^{(i)} s_i)$ for $j\in\{1,\dots,A\}$ and $G(1-a_j^{(i)} + \alpha_j^{(i)} s_i)$ for $j\in\{1,\dots,N_i\}$ and $\forall$ $i\in\{1,\dots,n\}$, lie to left of the contours.

An empty product is interpreted as unity. The poles of the integrand are assumed to be simple.

The integrals converge if $Re[s\log(z_i)-\log\sin\pi s]<0$ for large values of $|z_i|$ on the contours i.e. $|\{\arg(z_i)- W_1^{-1} W_2 \log|z_i|\}|<\pi$ for $i = 1,2,\dots,n$.

Finally, it is interesting to observe that the results (2.5),(3.1),(3.2) and (3.3) can be extended to a basic analogue of the H-function of several variables defined in this section.

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