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A STUDY OF THE DOUBLE LAPLACE TRANSFORM

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ABSTRACT

In this paper, we first establish three new theorems concerning two-dimensional Laplace transform. Next, we obtain an interesting and general integral from an application of the first theorem.

RESUMEN

En este artículo primero establecemos tres teoremas nuevos relacionados con la transformada bidimensional de Laplace. Luego obtenemos una integral general e interesante de una aplicación del primer teorema.

1. INTRODUCTION

A comprehensive account of two-dimensional Laplace transform has been given in the wellknown works of Voelker and Doetsch [11], Dirkin and Prudnikov [1]. The generalized form of Parseval-Goldstein theorem was given by Kalla [5] and later on this result was extended to domain of two variables by Bora (S.L.) and Saxena (R.K.).

In the present paper, we shall define and represent the two-dimensional Laplace transform in the following manner:

$$g(p,q) = p\,q \int_0^\infty \int_0^\infty e^{-px-qy}\, f(x,y)\, dx\, dy \tag{1.1}$$

where $Re(p) > 0$, $Re(q) > 0$ and the function $f(x,y)$ is so chosen that the above integral is absolutely convergent. The notation $g(p,q) \doteqdot f(x,y)$ will be used to denote (1.1) symbolically.

The H-function of two variables used in this paper is a special case of the general H-function of two variables studied earlier by Mittal and Gupta [7]. The parameters of this function occurring in the present paper will be displayed in the following contracted notation which is a direct extension to that of Srivastava and Joshi [10]:

$$H^{0,0\,:\,m_2,n_2\,;\,m_3,n_3}_{p_1,q_1\,:\,p_2,q_2\,;\,p_3,q_3}\left[\begin{matrix} x \\ y \end{matrix}\middle| \begin{matrix} (a_j;\alpha_j,A_j)_{1,p_1} : (c_j,\varepsilon_j)_{1,p_2} ; (e_j,E_j)_{1,p_3} \\ (b_j;\beta_j,B_j)_{1,q_1} : (d_j,\delta_j)_{1,q_2} ; (f_j,F_j)_{1,q_3} \end{matrix}\right]$$

$$= H_1\begin{bmatrix} x \\ y \end{bmatrix} = (2\pi i)^{-2} \int_{L_1}\int_{L_2} \phi(s,t)\, \theta_1(s)\, \theta_2(t)\, x^s\, y^t\, ds\, dt \tag{1.2}$$

where

$$\phi(s,t) = \left[\prod_{j=1}^{p_1} \Gamma(a_j - \alpha_j s - A_j t) \prod_{j=1}^{q_1} \Gamma(1 - b_j + \beta_j s + B_j t) \right]^{-1} \quad (1.3)$$

$$\theta_1(s) = \prod_{j=1}^{n_2} \Gamma(1 - c_j + \varepsilon_j s) \prod_{j=1}^{m_2} \Gamma(d_j - \delta_j s)$$

$$\times \left[\prod_{j=n_2+1}^{p_2} \Gamma(c_j - \varepsilon_j s) \prod_{j=m_2+1}^{q_2} \Gamma(1 - d_j + \delta_j s) \right]^{-1} \quad (1.4)$$

and with $\theta_2(t)$ defined analogously to $\theta_1(s)$ in terms of the parameter sets $(e_j, E_j)_{1,p_3}$ and $(f_j, F_j)_{1,q_3}$. Also, $(a_j ; \alpha_j , A_j)_{1,p_1}$ abbreviate the parameters sequence $(a_1 ; \alpha_1 , A_1)$, $(a_2 ; \alpha_2 , A_2), \ldots ,$ $(a_{p_1} ; \alpha_{p_1} , A_{p_1})$; $(c_j , \varepsilon_j)_{1,p_2}$ abbreviate the parameters sequence (c_1 , ε_1) , $(c_2 , \varepsilon_2), \ldots , (c_{p_2} , \varepsilon_{p_2})$ and so on.

The conditions on parameters of the H-function of two variables for the convergence of the integral given by (1.2) , the nature of contours L_1 and L_2 , some of the properties , special cases and asymptotic expansions of $H_1 \begin{bmatrix} x \\ y \end{bmatrix}$ can be referred to in the paper by Goyal [2]. It will be assumed that the conditions (i) to (vi), modified appropriately , given on p.119 in the paper by Mittal and Gupta [7] , are always satisfied by the various H-functions of two variables occurring in this paper.

To save space, three dots "..." appearing at a particular place in any H-function of two variables indicate that the parameters in that position are exactly same as those of the H-function of two variables defined by (1.2). Again $H_1\begin{bmatrix} Ax^u \\ By^v \end{bmatrix}$ will stand for the H-function of two variables defined by (1.2) but having arguments Ax^u, By^v instead of x,y.

2. MAIN THEOREMS

THEOREM 1. *If,*

$$f(p,q) \doteqdot g(x,y) \tag{2.1}$$

and

$$h(p,q) \doteqdot x^{\rho-2}\, y^{\sigma-2}\, H_1\begin{bmatrix} Ax^{-u} \\ By^{-v} \end{bmatrix} f(x,y) \tag{2.2}$$

then

$$h(p,q) = p\,q \int_0^\infty \int_0^\infty (x+p)^{-\rho}\,(y+q)^{-\sigma}\, g(x,y)$$

$$\times H^{0,0\,:\,m_2+1,n_2\,;\,m_3+1,n_3}_{p_1,q_1\,:\,p_2,q_2+1\,;\,p_3,q_3+1}\left[\begin{matrix} A(x+p)^u \\ B(y+q)^v \end{matrix}\,\middle|\,\begin{matrix} \ldots : & \ldots ; & \ldots \\ \ldots : (\rho,u), & \ldots ; (\sigma,v), & \ldots \end{matrix}\right] dx\, dy \tag{2.3}$$

where u and v are positive numbers, min $Re(p, q, \rho, \sigma) > 0$ *and the various integrals involved in* (2.1), (2.2) *and* (2.3) *are absolutely convergent.*

THEOREM 2. *If*

$$f(p,q) \doteqdot g(x,y) \tag{2.4}$$

and

$$h(p,q) \doteqdot x^{\rho-2} y^{\sigma-2} H_1 \begin{bmatrix} Ax^u \\ By^v \end{bmatrix} f(x,y) \tag{2.5}$$

then

$$h(p,q) = p\,q \int_0^\infty \int_0^\infty (x+p)^{-\rho} (y+q)^{-\sigma} g(x,y)$$

$$\times H^{0,0\,:\,m_2,n_2+1;m_3,n_3+1}_{p_1,q_1\,:\,p_2+1,q_2;p_3+1,q_3}\left[\begin{array}{l|l} A(x+p)^{-u} & \ldots:(1-\rho,u),\ldots;(1-\sigma,v),\ldots \\ B(y+q)^{-v} & \ldots: \qquad \ldots; \qquad \ldots \end{array}\right] dx\,dy \tag{2.6}$$

where u and v are positive numbers, min $Re(p, q, \rho, \sigma) > 0$ *and the various integrals involved in* (2.4) *to* (2.6) *are absolutely convergent.*

THEOREM 3. *If*

$$f(p,q) \doteqdot g(x,y) \tag{2.7}$$

and

$$h(p,q) \doteqdot x^{-\rho}\, y^{-\sigma}\, H_1 \begin{bmatrix} Ax^{u} \\ By^{v} \end{bmatrix} g(x,y) \tag{2.8}$$

then

$$h(p,q) = p\, q \int_0^\infty \int_0^\infty (x+p)^{-1} (y+q)^{-1} x^{\rho-1} y^{\sigma-1} f(x+p, y+q)$$

$$\times H^{0,0\,:\,m_2,n_2\,;\,m_3,n_3}_{p_1,q_1\,:\,p_2+1,q_2\,;\,p_3+1,q_3} \left[\begin{matrix} Ax^{-u} \\ By^{-v} \end{matrix} \middle| \begin{matrix} \ldots : \ldots, (\rho,u) ; \ldots, (\sigma,v) \\ \ldots : \quad \ldots ; \quad \ldots \end{matrix} \right] dx\, dy \tag{2.9}$$

where $u > 0$, $v > 0$, *min* $Re(p, q, \rho, \sigma) > 0$, *and the various integrals involved in* (2.7) *to* (2.9) *are absolutely convergent.*

To prove Theorem 1 , we first obtain the double Laplace transform of the function $x^{\rho-1}\, y^{\sigma-1}\, e^{-ax-by}\, H_1 \begin{bmatrix} Ax^{-u} \\ By^{-v} \end{bmatrix}$ which easily follows from the definitions (1.1) , (1.2) and a well known property of two-dimensional Laplace transform. The theorem now follows easily with the help of Parseval-Goldstein theorem for the double Laplace transform. The proofs of Theorem 2 and Theorem 3 can be developed on lines similar to those indicated above.

3. SPECIAL CASES

Since the H-function of two variables involved in all the above theorems is most general in nature, on making free use of its special cases as pointed out by Goyal [2, pp.121-125], one can easily obtain a large number of interesting and new theorems as special cases of these theorems.

Again proper specializations of Theorems 1, 2 and 3 readily yield analogous theorems for the wellknown one dimensional Laplace transform. It may be pointed out these theorems involving one dimensional Laplace transform are also new and sufficiently general in nature. Thus they generalize among others the theorems obtained earlier by Saxena [8, p.233] and Maloo [6, p.846]. We however do not record them here explicitly on account of the triviality of the analysis involved.

4. APPLICATIONS

Now we shall obtain a very general and interesting integral with the application of Theorem 1.

Take

$$g(x,y) = x^{\lambda-1}\, y^{\delta-1}\, H^{M_2,0}_{P_2,Q_2}\left[ax^{-h}\,\middle|\,\begin{matrix}(c_j',\varepsilon_j')_{1,P_2}\\(d_j',\delta_j')_{1,Q_2}\end{matrix}\right]$$

$$\times H^{M_3,0}_{P_3,Q_3}\left[bx^{-\mu}\,\middle|\,\begin{matrix}(e_j',E_j')_{1,P_3}\\(f_j',F_j')_{1,Q_3}\end{matrix}\right] \tag{4.1}$$

(where $H_{p,q}^{m,n}[x]$ is the well known Fox's H-function [3]) in (2.1), we get with the help of a known result [4, p.190],

$$f(p,q) = p^{-\lambda+1}\, q^{-\delta+1}\, H_{P_2,Q_2+1}^{M_2+1,\,0}\left[ap^{r}\,\middle|\,\begin{matrix}(c_j',\varepsilon_j')_{1,P_2}\\ (\lambda,r),\,(d_j',\delta_j')_{1,Q_2}\end{matrix}\right]$$

$$\times\, H_{P_3,Q_3+1}^{M_3+1,\,0}\left[bq^{\mu}\,\middle|\,\begin{matrix}(e_j',E_j')_{1,P_3}\\ (\delta,\mu),\,(f_j',F_j')_{1,Q_3}\end{matrix}\right] \tag{4.2}$$

where

$$U' = \sum_{j=1}^{M_2}\delta_j' - \sum_{j=M_2+1}^{Q_2}\delta_j' - \sum_{j=1}^{P_2}\varepsilon_j' > 0,\ |\arg a| < \frac{1}{2}U'\pi,$$

$$V' = \sum_{j=1}^{M_3}F_j' - \sum_{j=M_3+1}^{Q_3}F_j' - \sum_{j=1}^{P_3}E_j' > 0,\ |\arg b| < \frac{1}{2}V'\pi,$$

$Re(p) > 0$, $Re(q) > 0$, $Re(\lambda) > 0$, $Re(\delta) > 0$, $r,\mu > 0$.

Again, with the help of (2.2), (4.2) and (1.1), we easily get

$$h(p,q) = p\,q\int_0^\infty\int_0^\infty x^{\rho-\lambda-1}\,y^{\sigma-\delta-1}\,e^{-px-qy}\,H_1\left[\begin{matrix}Ax^{-u}\\ By^{-v}\end{matrix}\right]$$

$$\times H^{M_2+1,\,0}_{P_2,Q_2+1}\left[ax^{r}\,\middle|\,\begin{matrix}(c'_j,\varepsilon'_j)_{1,P_2}\\(\lambda,r),\,(d'_j,\delta'_j)_{1,Q_2}\end{matrix}\right]$$

$$\times H^{M_3+1,\,0}_{P_3,Q_3+1}\left[by^{\mu}\,\middle|\,\begin{matrix}(e'_j,E'_j)_{1,P_3}\\(\delta,\mu),\,(f'_j,F'_j)_{1,Q_3}\end{matrix}\right]dx\,dy \qquad (4.3)$$

To evaluate the double integral on the right-hand side of (4.3), expand e^{-px-qy} in double series, interchange the order of integration and summation (which is justified under the conditions stated below), and evaluate the (x,y)-integral thus obtained with the help of a recent result of Singhal and Bhati [9, p.74, Eq.(1.1)], we obtain

$$h(p,q) = \frac{pq}{r\mu}\,a^{\frac{\lambda-\rho}{r}}\,b^{\frac{\delta-\sigma}{\mu}}\sum_{m,n=0}^{\infty}\frac{\left(pa^{-\frac{1}{r}}\right)^m\left(qb^{-\frac{1}{\mu}}\right)^n}{m!\;n!}$$

$$\times H^{0,\,0\,:\,m_2+M_2+1,\,n_2\,;\,m_3+M_3+1,\,n_3}_{p_1,q_1\,:\,p_2+P_2,\,q_2+Q_2+1\,;\,p_3+P_3,\,q_3+Q_3+1}\left[\begin{matrix}Aa^{\frac{u}{r}}\\Bb^{\frac{v}{\mu}}\end{matrix}\,\middle|\,\begin{matrix}\Phi\\\Psi\end{matrix}\right] \qquad (4.4)$$

where $\Phi = \left(c'_j+\frac{\rho-\lambda+m}{r}\varepsilon'_j,\,\frac{u}{r}\varepsilon'_j\right)_{1,P_2};\,\ldots,\left(e'_j+\frac{\sigma-\delta+n}{\mu}E'_j,\,\frac{v}{\mu}E'_j\right)_{1,P_3};$

$\Psi = (\rho+m,u),\,(d_j,\delta_j)_{1,m_2},\left(d'_j+\frac{\rho-\lambda+m}{r}\delta'_j,\,\frac{u}{r}\delta'_j\right)_{1,Q_2},\,(d_j,\delta_j)_{m_2+1,q_2};$

$(\sigma+n,v),\,(f_j,F_j)_{1,m_3},\left(f'_j+\frac{\sigma-\delta+n}{\mu}F'_j,\,\frac{v}{\mu}F'_j\right)_{1,Q_3},\,(f_j,F_j)_{m_3+1,q_3},$

$$\begin{matrix} m_3+M_3+1\ ,\ n_3 \\ p_3+P_3+1\ ,\ q_3+Q_3+1 \end{matrix}\left[\begin{matrix} Aa^{\frac{u}{h}} \\ Bb^{\frac{v}{\mu}} \end{matrix}\middle|\begin{matrix} \xi \\ \eta \end{matrix}\right] \tag{4.5}$$

where $\xi = \left(c_j' + \frac{\rho-\lambda+m}{h}\varepsilon_j',\ \frac{u}{h}\varepsilon_j'\right)_{1,P_2},\ (\rho,u)\ ;\ \ldots,$

$\left(e_j' + \frac{\sigma-\delta+n}{\mu}E_j',\ \frac{v}{\mu}E_j'\right)_{1,P_3},\ (\sigma,v)\ ;$

$\eta = (\rho+m,u),\ (d_j,\delta_j)_{1,m_2},\ \left(d_j' + \frac{\rho-\lambda+m}{h}\delta_j',\ \frac{u}{h}\delta_j'\right)_{1,Q_2},\ (d_j,\delta_j)_{m_2+1,q_2};$

$(\sigma+n,v),\ (f_j,F_j)_{1,m_3},\ \left(f_j' + \frac{\sigma-\delta+n}{\mu}F_j',\ \frac{v}{\mu}F_j'\right)_{1,Q_3},\ (f_j,F_j)_{m_3+1,q_3},$

where $Re(\lambda) > 0$, $Re(\delta) > 0$, the parameters of $H_1\left[\begin{matrix} A(x+p)^u \\ B(y+q)^v \end{matrix}\right]$ are same as given by (1.2) and the set of conditions mentioned with (4.4) are satisfied.

The integral (4.5) is believed to be one of the most general double integral evaluated so far . It is capable of yielding a large number of known as well as new integrals as its particular cases. Thus on making free use of the various formulae given by Goyal [2, pp.121-125] , Gupta and Jain [3, pp.598-601] , one can easily obtain from this integral , a number of other interesting integrals concerning a large spectrum of special functions . On proceeding in a manner similar to that of obtaining the integral (4.5) , we can also obtain two more integrals from theorems 2 and 3 . We however do not record them here on account of lack of space.

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provided that $Re(p) > 0$, $Re(q) > 0$, $u,v,r,\mu > 0$, $U' > 0$, $V' > 0$, $|arg\ a| < \frac{1}{2} U'\pi$, $|arg\ b| < \frac{1}{2} V'\pi$, $Re\left(\rho - u\ max\left\{(c_j - 1)/\varepsilon_j\right\}\right) > 0$, $Re\left(\sigma - v\ max\left\{(e_i - 1)/E_i\right\}\right) > 0$ $(j = 1, \ldots, n_2\ ;\ i = 1, \ldots, m_3)$, $Re\left(\rho - \lambda + r\ min\ (d'_i/\delta'_i) - u\ max\left\{(c_j - 1)/\varepsilon_j\right\}\right) > 0$ $(i = 1, \ldots, M_2\ ;$ $j = 1, \ldots, n_2)$, $Re\left(\sigma - \delta + \mu\ min(f_i/F_i) - v\ max\left\{(e_j - 1)/E_j\right\}\right) > 0$ $(i = 1, \ldots, M_3\ ;\ j = 1, \ldots n_3)$ and the double series on the right-hand side is absolutely convergent.

Substituting these values of $g(x,y)$ and $h(p,q)$ from (4.1) and (4.4) respectively in (2.3), altering the parameters of the H-function of two variables occurring on the right-hand side of (2.3) slightly, we arrive at the following interesting integral after a little simplification:

$$\int_0^\infty \int_0^\infty x^{\lambda-1}\ y^{\delta-1}\ (x+p)^{-\rho}\ (y+q)^{-\sigma}$$

$$\times\ H_1\begin{bmatrix} A(x+p)^u \\ B(y+q)^v \end{bmatrix}\ H^{M_2,0}_{P_2,Q_2}\left[ax^{-r}\ \middle|\ \begin{matrix}(c'_j, \varepsilon'_j)_{1,P_2} \\ (d'_j, \delta'_j)_{1,Q_2}\end{matrix}\right]$$

$$\times\ H^{M_3,0}_{P_3,Q_3}\left[by^{-\mu}\ \middle|\ \begin{matrix}(e'_j, E'_j)_{1,P_3} \\ (f'_j, F'_j)_{1,Q_3}\end{matrix}\right] dx\ dy$$

$$= \frac{a^{\frac{\lambda-\rho}{r}}\ b^{\frac{\delta-\sigma}{\mu}}}{r\ \mu} \sum_{m,n=0}^{\infty} \frac{\left(pa^{-\frac{1}{r}}\right)^m \left(qb^{-\frac{1}{\mu}}\right)^n}{m!\ n!}\ H^{0,0\ :\ m_2+M_2+1\ ,\ n_2\ ;}_{p_1,q_1\ :\ p_2+P_2+1\ ,\ q_2+Q_2+1\ ;}$$

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