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CONVERGENCE PROPERTIES AND AN INVERSION FORMULA FOR THE LAMBERT TRANSFORM

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ABSTRACT

The object of the present paper is to discuss certain convergence properties of the Lambert transform and to develop an inversion technique . The properties given here are necessary in the application of this transform to integral equations, differential equations and methods of analytic continuation of power series.

RESUMEN

El objeto de este trabajo es discutir algunas propiedades de convergencia de la transformada de Lambert, y desarrollar una técnica de inversión. Las propiedades dadas son necesarias para la aplicación de esta transformada a ecuaciones diferenciales , ecuaciones integrales y los métodos de continuación analítica de series de potencias.

1. INTRODUCTION

In this paper certain convergence properties of the Lambert transform are discussed and an inversion technique is developed. The convergence properties discussed herein are essential in the determination of the analyticity of the Lambert transform and these properties are necessary in the application of the transform to integral equations, differential equations, or methods of analytic continuation of power series.

If we denote the Lambert transform of $F(t)$ by

$$LM\{F(t)\} = \int_0^\infty \frac{st}{e^{st}-1} F(t)\,dt = f(s)$$

for those values of s for which the integral converges, the following can easily be verified using tables of integral transforms or by writing $\frac{1}{e^{st}-1} = \sum_{n=1}^{\infty} e^{-nst}$.

$$LM\{t^\alpha\} = \frac{\Gamma(\alpha+2)\ \zeta(\alpha+2)}{s^{\alpha+1}}, \quad Re\ s > 0, \quad Re\ \alpha > -1, \qquad (1.1)$$

where $\Gamma(z)$ is the gamma function and $\zeta(z)$ is the Riemann zeta function defined as

$$\zeta(z) = \sum_{k=1}^{\infty} \frac{1}{k^z}$$

It can be observed once the convergence properties have been established that the Lambert transform will exist for the class of functions $\mathfrak{C}$ having the property that $F \varepsilon \mathfrak{C}$ implies $F(z)$ has the expansion $F(z) = \sum_{n=0}^{\infty} a_n z^{\lambda n}$, where $-1 < \lambda_0 < \lambda_1 < \ldots$ is an increasing sequence of real numbers with $\lambda_n \to +\infty$ as $n \to \infty$ and the series for F has a non-zero radius of convergence.

2. CONVERGENCE OF THE LAMBERT TRANSFORM

Goldberg [4] proved that if the transform

$$F(x) = \int_{0^+}^{\infty} K(x,t)\, d\,\alpha(t), \quad x \text{ real}, \tag{2.1}$$

$K(x,t) = \sum_{k=1}^{\infty} a_k\, e^{-kxt}$ for fairly general class of sequences $\{a_k\}$, converges for some $x_0 > 0$ and if $a_k = O(k^{r-1})$ for some $r \geq 0$, as $k \to \infty$, and

$$\int_0^1 \frac{|\alpha(t)|}{t^{r+1}}\, dt < \infty ,$$

then the transform (2.1) converges for all $x \geq x_0$. This result will be extended in this section.

In discussing convergence it can be seen that the Lambert transform of the function $f(t) = t^{-1}$, $t > 0$, fails to exist due to the behavior of the function as $t \to 0^+$. However, the Lambert transform of $G(t) = t^{-\frac{1}{2}}$, $t > 0$, exists and has the value $LM\{t^{-\frac{1}{2}}\} = \dfrac{\Gamma\left(\frac{3}{2}\right)\zeta\left(\frac{3}{2}\right)}{s^{\frac{1}{2}}}$ provided $Re\, s > 0$. In fact, convergence of the transform at the lower limit is assured for $F(t)$ if $t^{1-\delta}F(t) \to 0$ as $t \to 0^+$ for some $\delta \in (0,1)$. An equivalent way of assuring convergence at the lower limit of the transform, as is common in the discussion of convergence of the Laplace transform, is to restrict our class of functions to those functions $F(t)$ such that $\int_0^b |F(t)|dt$ is convergent with respect to the lower limit.

THEOREM 2.1 *If the Lambert integral*

$$LM\{F(t)\} = \int_0^\infty \frac{st}{e^{st}-1} F(t)dt \tag{2.2}$$

is convergent for $s = s_0$ with $Re\ s_0 > 0$, then the integral is convergent for all s with $Re\ s > Re\ s_0$.

Proof: First we consider the integral (2.2) with respect to its lower limit. Since we are considering functions $F(t)$ such that $\int_0^b |F(t)|dt$ converges with respect to the lower limit, then given $\varepsilon > 0$, there is a $\delta > 0$ such that for all δ_1, δ_2 with $0 < \delta_1 < \delta_2 < \delta$,

$$\int_{\delta_1}^{\delta_2} |F(t)|dt < \varepsilon/2 .$$

It can be shown that for all s with $Re\ s \neq 0$

$$\lim_{t \to 0} \left|\frac{st}{e^{st}-1}\right| = 1 .$$

This implies the existance of a real number $r > 0$ such that for all $t \in (0,r)$,

$$\left|\frac{st}{e^{st}-1}\right| < 2 .$$

Hence, if $0 < \delta_1 < \delta_2 < min\ (\delta,r)$, we have

$$\left|\int_{\delta_1}^{\delta_2} \frac{st}{e^{st}-1} F(t)dt\right| \leq \int_{\delta_1}^{\delta_2} \left|\frac{st}{e^{st}-1}\right| |F(t)|dt <$$

$$< 2 \int_{\delta_1}^{\delta_2} |F(t)|dt < \varepsilon$$

which implies convergence at the lower limit of (2.2) for all s with $Re\, s \neq 0$, and in particular for $Re\, s > Re\, s_0$.

In consideration of the upper limit, suppose $LM\{F(t)\}$ is convergent for $s = s_0$ and set

$$R(x) = \int_x^{\infty} \frac{s_0 t}{e^{s_0 t}-1} F(t)dt \tag{2.3}$$

so that $|R(x)|$ can be made arbitrarily small by choosing x sufficiently large. We also have, for almost all x,

$$R'(x) = -\frac{s_0 x}{e^{s_0 x}-1} F(x) .$$

Now consider for $w > x$,

$$\int_x^w \frac{st}{e^{st}-1} F(t)dt = \int_x^w \frac{s(e^{s_0 t}-1)}{s_0(e^{st}-1)} \frac{s_0 t}{e^{s_0 t}-1} F(t)dt$$

$$= -\frac{s}{s_0} \int_x^w \frac{e^{s_0 t}-1}{e^{st}-1} R'(t)dt .$$

Performing integration by parts we have

$$\int_x^w \frac{st}{e^{st}-1} F(t)dt = -\frac{s}{s_0} \frac{e^{s_0 t}-1}{e^{st}-1} R(t)\Big|_x^w$$

$$+ \frac{s}{s_0} \int_x^w \left[\frac{s_0 e^{s_0 t}}{e^{st}-1} - \frac{se^{st}(e^{s_0 t}-1)}{(e^{st}-1)^2}\right] R(t)dt$$

$$= \frac{s}{s_0}\frac{e^{s_0 x}-1}{e^{sx}-1}R(x) - \frac{s}{s_0}\frac{e^{s_0 w}-1}{e^{sw}-1}R(w)$$

$$+ \int_x^w \frac{s e^{s_0 t}R(t)}{e^{st}-1}\,dt - \int_x^w \frac{s^2 e^{st}(e^{s_0 t}-1)}{s_0(e^{st}-1)^2}R(t)\,dt . \qquad (2.4)$$

By hypothesis we assume $Re\ s_0 > 0$ so that for an arbitrary but fixed s with $Re\ s > Re\ s_0 > 0$ we have

$$\lim_{t\to\infty}\left|\frac{s}{s_0}\frac{e^{s_0 t}-1}{e^{st}-1}\right| \leq \left|\frac{s}{s_0}\right|\lim_{t\to\infty}\frac{e^{Re\ s_0 t}+1}{e^{Re\ st}-1} = 0 ,$$

which implies that we can find a real number x_1 such that for all $t > x_1$,

$$\left|\frac{s}{s_0}\frac{e^{s_0 t}-1}{e^{st}-1}\right| < 1 .$$

Since $|R(t)|$ can be made arbitrarily small by choosing t sufficiently large , then for all t larger than some number $x_2 \geq x_1$ we have

$$\left|\frac{s}{s_0}\frac{e^{s_0 t}-1}{e^{st}-1}R(t)\right| < |R(t)| < \frac{\varepsilon}{4} . \qquad (2.5)$$

Also for s fixed with $Re\ s > Re\ s_0 > 0$, it is possible to find a real number $\eta > 0$ such that $Re\ s > Re\ s_0 + \eta$. Thus we can write

$$\left|\frac{s e^{s_0 t}}{e^{st}-1}\right| = \left|\frac{s e^{(s_0+\eta)t}}{e^{st}-1}\right| e^{-\eta t}$$

and

$$\lim_{t\to\infty}\left|\frac{se^{(s_0+\eta)t}}{e^{st}-1}\right| \le |s| \lim_{t\to\infty}\frac{e^{Re(s_0+\eta)t}}{e^{Re\, st}-1}$$

$$= \frac{|s|}{Re\, s} Re(s_0+\eta) \lim_{t\to\infty} e^{-Re(s-s_0-\eta)t} = 0 ,$$

since $Re\, s > Re\, s_0 > 0$ and $Re(s-s_0-\eta) > 0$. It is then possible to find a real number x_3 sufficiently large so that for all $t > x_3$ we have the following inequalities;

$$\left|\frac{se^{(s_0+\eta)t}}{e^{st}-1}\right| < 1 , \quad |R(t)| < \frac{\varepsilon\eta}{4} .$$

Thus for all x and w with $x_3 < x < w$,

$$\left|\int_x^w \frac{se^{s_0 t}}{e^{st}-1} R(t)dt\right| \le \int_x^w \left|\frac{se^{(s_0+\eta)t}}{e^{st}-1}\right| |R(t)|\, e^{-\eta t}dt$$

$$\le \frac{\varepsilon\eta}{4}\int_x^w e^{-\eta t}dt < \frac{\varepsilon\eta}{4}\int_0^\infty e^{-\eta t}dt = \frac{\varepsilon}{4} .$$

(2.6)

Again considering s arbitrary but fixed with $Re\, s > Re\, s_0 > 0$ we choose $\eta > 0$ such that $Re\, s > Re\, s_0 + \eta$. Then

$$\left|\frac{s^2 e^{st}(e^{s_0 t}-1)}{s_0(e^{st}-1)^2}\right| = \left|\frac{e^{st}}{e^{st}-1}\right| \left|\frac{s^2}{s_0}\,\frac{e^{s_0 t}-1}{e^{st}-1}\right|$$

$$= \left|\frac{1}{1-e^{-st}}\right| \left|\frac{s^2(e^{s_0 t}-1)e^{\eta t}}{s_0(e^{st}-1)}\right| e^{-\eta t}$$

(2.7)

and we may write

$$\left|\frac{s^2(e^{s_0 t}-1)e^{\eta t}}{s_0(e^{st}-1)}\right| = \left|\frac{s^2}{s_0}\frac{e^{(s_0+\eta)t}}{e^{st}-1} - \frac{s^2 e^{\eta t}}{s_0(e^{st}-1)}\right|$$

$$\leq \left|\frac{s^2 e^{(s_0+\eta)t}}{s_0(e^{st}-1)}\right| + \left|\frac{s^2 e^{\eta t}}{s_0(e^{st}-1)}\right| . \qquad (2.8)$$

However,

$$\lim_{t\to\infty}\left|\frac{s^2}{s_0}\frac{e^{(s_0+\eta)t}}{e^{st}-1}\right| \leq \left|\frac{s^2}{s_0}\right| \lim_{t\to\infty}\frac{e^{Re(s_0+\eta)t}}{e^{Re\, st}-1}$$

$$\leq \left|\frac{s^2}{s_0}\right| \frac{Re(s_0+\eta)}{Re\, s} \lim_{t\to\infty} e^{-Re(s-s_0-\eta)t} = 0,$$

since $Re(s-s_0-\eta) > 0$, and we also have

$$\lim_{t\to\infty}\left|\frac{s^2}{s_0}\frac{e^{\eta t}}{e^{st}-1}\right| \leq \left|\frac{s^2}{s_0}\right| \lim_{t\to\infty}\frac{e^{\eta t}}{e^{Re\, st}-1}$$

$$= \left|\frac{s^2}{s_0}\right| \frac{\eta}{Re\, s} \lim_{t\to\infty} e^{-Re(s-\eta)t} = 0$$

since $Re\, s > Re\, s_0 + \eta$ implies $Re(s-\eta) > Re\, s_0 > 0$. Also

$$\lim_{t\to\infty}\left|\frac{1}{1-e^{-st}}\right| \leq \lim_{t\to\infty}\frac{1}{1-e^{-Re\, st}} = 1$$

since $Re\, s > 0$. Accordingly we may find an x_4 sufficiently large so that each of the following inequalities is valid for all $t > x_4$;

$$\left|\frac{s^2}{s_0}\frac{e^{(s_0+\eta)t}}{e^{st}-1}\right| \leq 1/4 , \qquad (2.9)$$

$$\left|\frac{s^2}{s_0}\frac{e^{\eta t}}{e^{st}-1}\right| < 1/4 , \qquad (2.10)$$

$$\left|\frac{1}{1-e^{-st}}\right| < 2 , \qquad (2.11)$$

$$|R(t)| < \frac{\varepsilon\eta}{4} . \qquad (2.12)$$

Thus , applying (2.9) and (2.10) to (2.8) and this result along with (2.11) to (2.7), we may conclude that for s fixed with $Re\ s > Re\ s_0 > 0$ and choosing $\eta > 0$ such that $Re\ s > Re\ s_0 + \eta$, then for all $t > x_4$ we have

$$\left|\frac{s^2 e^{st}(e^{s_0 t}-1)}{s_0 (e^{st}-1)^2}\right| < e^{-\eta t} ,$$

so that using this result and that of (2.12) , then for all x and w with $x_4 < x < w$,

$$\left|\int_x^w \frac{s^2 e^{st}(e^{s_0 t}-1)}{s_0 (e^{st}-1)^2} R(t)\,dt\right| \leq \int_x^w \left|\frac{s^2 e^{st}(e^{s_0 t}-1)}{s_0 (e^{st}-1)^2}\right| |R(t)|\,dt$$

$$< \frac{\varepsilon\eta}{4}\int_x^w e^{-\eta t}dt < \frac{\varepsilon\eta}{4}\int_0^\infty e^{-\eta t}dt = \frac{\varepsilon}{4} .$$

(2.13)

Hence , taking the absolute value on both sides of (2.4) , using the triangle inequality on the right-hand side and refering to (2.5) , (2.6) and (2.13) we can conclude that given s, arbitrary but fixed, with $Re\, s > Re\, s_0 > 0$, and given $\varepsilon > 0$ we can find a $K = max\ (x_2, x_3, x_4)$ so that for all x and w with $K < x < w$,

$$\left| \int_x^w \frac{st}{e^{st}-1} F(t)dt \right| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon .$$

This concludes the proof of the theorem.

It can be noted that if we restrict s to a bounded region B properly contained in the region $Re\ (s-s_0) > 0$ we can arrive at inequalities independent of any $s \in B$. This leads to the following theorem.

THEOREM 2.2 *If the Lambert integral*

$$LM\{F(t)\} = \int_0^\infty \frac{st}{e^{st}-1} F(t)dt$$

is convergent at $s = s_0$ *where* $Re\ s_0 > 0$, *then for an arbitrary real number* $\eta > 0$ *the Lambert integral is uniformly convergent throughout any bounded region contained in the half plane* $Re\ (s - s_0) \geq \eta > 0$.

3. AN INVERSION FORMULA

We now state as a lemma a Cauchy integral type formula that will be used to develop an inversion technique which is very useful for a certain class of functions. This lemma will be proved for completeness.

LEMMA 3.1 *Let $f(z)$ be of the form $f(z) = \sum_{n=1}^{\infty} \frac{a_n}{z^n}$, converging for complex z with $|z| > R > 0$, the a_n being constants. Then, letting C denote the circle $z = \rho e^{i\theta}$, $\rho > R$, $0 \leq \theta < 2\pi$, the formula*

$$\frac{1}{2\pi i} \int_C \frac{f(z)}{a - z} dz = f(a) \tag{3.1}$$

is valid for all complex values a with $|a| > \rho > R$.

Proof: The integral in (3.1) is familiar in the derivation of Laurent series. Here we evaluate the integral by making the substitution $z = \frac{1}{w}$, which yields

$$\frac{1}{2\pi i} \int_C \frac{f(z)}{a - z} dz = \frac{1}{a} \cdot \frac{1}{2\pi i} \int_{c_1^+} \frac{g(w)}{w - \frac{1}{a}} dw, \tag{3.2}$$

where $g(w) = \frac{1}{w} f(\frac{1}{w}) = a_1 + a_2 w + \ldots$, which by hypothesis will converge for $|w| < \frac{1}{R} = r$, and where c_1 is the circle $w = r e^{-i\theta}$, $r = \frac{1}{R}$, $0 \leq \theta \leq 2\pi$. Now, applying the Cauchy integral formula, we have, since $\frac{1}{a}$ is interior to c_1,

$$\frac{1}{2\pi i} \int_{c_1^+} \frac{g(w)}{w - \frac{1}{a}} dw = g(\frac{1}{a}) = a f(a). \tag{3.3}$$

Substituting the result of (3.3) into (3.2) we have

$$\frac{1}{2\pi i} \int_C \frac{f(z)}{a - z} dz = \frac{1}{a} [a f(a)] = f(a),$$

the desired result.

We will now develop an inversion formula for the Lambert transform which is accomplished by use of the Residue Theorem.

THEOREM 3.1 *If*

$$f(s) = \int_0^\infty \frac{st}{e^{st}-1} F(t)dt \tag{3.4}$$

and if the generating function can be written in the form

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{s^n}, \quad |s| > R,$$

the integral in (3.4) *converging for at least* $Re\ s > \rho$, *with* $\rho > R$, *then, letting*

$$\phi(zt) = \sum_{n=0}^{\infty} \frac{(zt)^n}{\Gamma(n+2)\ \zeta(n+2)}$$

and c *be the curve* $z = r\ e^{i\theta}$, $0 \leq \theta \leq 2\pi$, $r > \rho$, *we have*

$$F(t) = \frac{1}{2\pi i} \int_c \phi(zt)\ f(z)\ dz$$

$$= - \operatorname*{Res}_{z=\infty} \phi(zt)\ f(z). \tag{3.5}$$

Proof: Using the function ϕ and f and the curve c described in the hypothesis, we consider

$$\int_0^\infty \frac{st}{e^{st}-1} [\frac{1}{2\pi i} \int_c \phi(zt)\ f(z)\ dz]\ dt. \tag{3.6}$$

For fixed s in the region $A = \{s : Re\, s \neq 0\}$, let

$$G(z,t) = K(s,t)\, \phi(zt)\, f(z),$$

where as always $K(s,t) = \frac{st}{e^{st}-1}$, $0 < t < \infty$, and $K(s,0) = 1$. We note that for each fixed t, $0 \leq t < \infty$, the function $G(z,t)$ will be continuous with respect to z along the curve c since $\phi(zt)$ is an entire function and $f(z)$ is analytic for $|z| = r > R$. For each fixed z on c, the function $G(z,t)$ is continuous with respect to t on any interval $[a,b]$, $0 \leq a < b < \infty$, since $\phi(zt)$ is entire and $K(s,t)$ is continuous with respect to t. Also, we have

$$\int_0^\infty \frac{st}{e^{st}-1}\, \phi(zt)\, f(z)\, dt = f(z) \int_0^\infty \frac{st}{e^{st}-1} \sum_{n=0}^{\infty} \frac{(zt)^n}{(n+1)!\, \zeta(n+2)}\, dt. \tag{3.7}$$

Now the series $\phi(zt) = \sum_{n=0}^{\infty} \frac{(zt)^n}{(n+1)!\, \zeta(n+2)}$ is entire in zt and hence is uniformly convergent as regards t in the interval $0 \leq t \leq w$, $w < \infty$. Hence,

$$\int_0^w \frac{st}{e^{st}-1} \sum_{n=0}^{\infty} \frac{(zt)^n}{(n+1)!\, \zeta(n+2)}\, dt = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!\, \zeta(n+2)} \int_0^w \frac{st}{e^{st}-1}\, t^n\, dt.$$

But the series on the right is shown to be uniformly convergent as regards w as follows:

$$\left| \int_0^w \frac{st}{e^{st}-1}\, t^n\, dt \right| \leq \int_0^w \frac{|s|\, t^{n+1}}{|e^{st}-1|}\, dt\ ,$$

and if $Re\, s > 0$, $|e^{st}-1| \geq e^{(Re s)t} - 1$, $t > 0$, so that

$$\int_0^w \frac{|s|\ t^{n+1}}{|e^{st}-1|}\,dt \le |s| \int_0^w \frac{t^{n+1}}{e^{(Re s)t}-1}\,dt < |s| \int_0^\infty \frac{t^{n+1}}{e^{(Re s)t}-1}\,dt\ .$$

This last integral converges since in the hypothesis we have assumed $Re\, s > \rho > R > 0$.

We can therefore let $w \to \infty$, and provided z satisfies $|z| < |s|$, we arrive at

$$\int_0^\infty \frac{st}{e^{st}-1} \sum_{n=0}^\infty \frac{(zt)^n}{(n+1)!\ \zeta(n+2)}\,dt = \sum_{n=0}^\infty \frac{z^n}{(n+1)!\ \zeta(n+2)} \int_0^\infty \frac{st}{e^{st}-1}\, t^n\,dt$$

$$= \sum_{n=0}^\infty \frac{z^n}{s^{n+1}} = \frac{1}{s - z}$$

Thus we can conclude that for any $\delta > 1$, the integral of (3.7) converges uniformly to $\frac{f(z)}{s - z}$ for z on c and s in the region $A_\delta = \{s : Re\, s > 0,\ |s| \ge \delta|z|\}$.

The conditions just discussed are precisely the conditions in the hypothesis of Weierstrass's Theorem [7, p. 97] on interchanging the order of integration. Thus, interchanging the order of integration in (3.6)

$$\int_0^\infty \frac{st}{e^{st}-1} \left[\frac{1}{2\pi i} \int_c \phi(zt)\ f(z)dz\right] dt = \frac{1}{2\pi i} \int_c f(z)\left[\int_0 \frac{st}{e^{st}-1}\,\phi(zt)dt\right] dz$$

$$= \frac{1}{2\pi i} \int_c \frac{f(z)}{s - z}\,dz\ ,$$

provided $|s| > |z| = r$. However, for $|s| > r > R$, the conditions of Lemma 1 are satisfied and we can conclude that for all s with $Re\, s > r$ this last integral has the desired value of $f(s)$. It is then evident that if $f(s)$ satisfies the hypothesis of the theorem, then for almost all t with $0 < t < \infty$,

$$F(t) = \frac{1}{2\pi i} \int_c \phi(zt)\, f(z)dz .$$

We note that since $r > R$ then all finite singularities of $f(s)$ are interior to c and according to a consequence of the Residue Theorem, for almost all t, $0 < t < \infty$,

$$F(t) = \frac{1}{2\pi i} \int_c \phi(zt)\, f(z)\, dz = - \operatorname*{Res}_{z=\infty} \phi(zt)\, f(z) ,$$

since $\phi(zt)$ is an entire function, and the theorem is proved.

The previous theorem furnishes a convenient means of establishing the following formula.

COROLLARY 3.1 *If the function $f(s)$ can be represented in a series of the form*

$$f(s) = \sum_{n=0}^{\infty} \frac{a_n}{s^{n+1}},$$

converging for $|s| > a \geq 0$, $a < \infty$, then the determining function

$$F(t) = LM^{-1} \{f(s)\}$$

can be computed through term by term inversion. That is,

$$F(t) = \sum_{n=0}^{\infty} a_n\, LM^{-1} \{\frac{1}{s^{n+1}}\} = \sum_{n=0}^{\infty} \frac{a_n\, t^n}{(n+1)!\, \zeta(n+2)} ,$$

this series for $F(t)$ converging for all t.

Proof: If $f(s)$ has the form given in the hypothesis, then from Theorem 3.1 we have

$$F(t) = - \operatorname*{Res}_{z=\infty} \phi(zt)\, f(z),$$

where $\phi(zt) = \sum_{n=0}^{\infty} \frac{(zt)^n}{(n+1)!\,\zeta(n+2)}$. However, $- \operatorname*{Res}_{z=\infty} \phi(zt)\, f(z) =$ $\operatorname*{Res}_{z=\infty} \frac{1}{z^2}\,\phi(\frac{t}{z})\, f(\frac{1}{z})$, and

$$\frac{1}{z^2}\,\phi(\frac{t}{z})\, f(\frac{1}{z}) = \frac{1}{z^2} \sum_{k=0}^{\infty} \frac{t^k}{(k+1)!\,\zeta(k+2) z^k} \sum_{\nu=0}^{\infty} a_\nu z^{\nu+1}$$

$$= \sum_{k=0}^{\infty} \frac{t^k}{(k+1)!\,\zeta(k+2) z^{k+2}} \sum_{\nu=0}^{\infty} a_\nu z^{\nu+1}.$$

The coefficient of $\frac{1}{z}$ in the resulting Laurent series is

$$\sum_{n=0}^{\infty} \frac{a_n t^n}{(n+1)!\,\zeta(n+2)},$$

which proves that $F(t)$ can be represented by the desired series. The convergence of this series for $F(t)$ for all t is evident since by hypothesis the series $\sum_{n=0}^{\infty} a_n w^{n+1}$ has a non-zero radius of convergence.

As an example consider the function $f(s) = \frac{1}{s-a}$. Expanding $f(s)$ about $s = 0$, we have

$$f(s) = \frac{1}{s-a} = \frac{1}{s}\,\frac{1}{1-\frac{a}{s}} = \sum_{n=0}^{\infty} \frac{a^n}{s^{n+1}},$$

converging for $|s| > |a|$. Then

$$F(t) = LM^{-1}\left\{\frac{1}{s-a}\right\} = \sum_{n=0}^{\infty} \frac{a^n t^n}{(n+1)!\,\zeta(n+2)} \quad .$$

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